

# A $W_2^n$ -Theory of Stochastic Parabolic Partial Differential Systems on $C^1$ -domains

(running title: SPDSs on  $C^1$ -domains)

Kyeong-Hun Kim\* and Kijung Lee†

## Abstract

In this article we present a  $W_2^n$ -theory of stochastic parabolic partial differential systems. In particular, we focus on non-divergent type. The space domains we consider are  $\mathbb{R}^d$ ,  $\mathbb{R}_+^d$  and eventually general bounded  $C^1$ -domains  $\mathcal{O}$ . By the nature of stochastic parabolic equations we need weighted Sobolev spaces to prove the existence and the uniqueness. In our choice of spaces we allow the derivatives of the solution to blow up near the boundary and moreover the coefficients of the systems are allowed to oscillate to a great extent or blow up near the boundary.

*Keywords:* stochastic parabolic partial differential systems, weighted Sobolev spaces.

*AMS 2000 subject classifications:* primary 60H15, 35R60; secondary 35K45, 35K50.

## 1 Introduction

In this article we consider the following general stochastic parabolic partial differential system :

$$\begin{aligned} du^k &= (a_{kr}^{ij} u_{x^i x^j}^r + b_{kr}^i u_{x^i}^r + c_{kr} u^r + f^k) dt \\ &\quad + (\sigma_{kr,m}^i u_{x^i}^r + \nu_{kr,m} u^r + g_m^k) dw_t^m, \quad t > 0, x \in \mathcal{O} \subset \mathbb{R}^d \\ u^k(0) &= u_0^k, \end{aligned} \tag{1.1}$$

where  $i, j = 1, 2, \dots, d$  and  $k, r = 1, 2, \dots, d_1$  and we used the summation convention on the repeated indices  $i, j, r$ . The system (1.1) models the interactions among  $d_1$  diffusive quantities with other physical phenomena like convection, internal source or sink, and randomness caused by lack of information. Moreover, the countable sum of the stochastic integrals against independent one-dimensional Brownian motions  $\{w^m : m = 1, 2, \dots\}$  enables us to include the stochastic integral against a cylindrical Brownian motion in (1.1) (see sec. 8.2 of [12]). The solution  $u = (u_1, u_2, \dots, u_{d_1})$  not only depends on  $t > 0$ ,  $x \in \mathcal{O}$ , but also depends on  $\omega$  in a probability space  $(\Omega, \mathcal{F}, \{\mathcal{F}_t; t \geq 0\}, P)$  on which  $w^m$  are defined. The coefficients  $a_{kr}^{ij}, b_{kr}^i, c_{kr}, \sigma_{kr,m}^i, \nu_{kr,m}$  also depend on  $(\omega, t, x)$ . The detailed formulation of (1.1) follows in the subsequent sections.

---

\*Department of Mathematics, Korea University, Seoul, South Korea 136-701, kyeonghun@korea.ac.kr.

†Department of Mathematics, Ajou University, Suwon, South Korea 443-749, kijung@ajou.ac.kr.

The concrete motivations of studying (1.1) can be easily found in the literature. If  $d_1 = 1$ , (1.1) is a stochastic partial differential equation (SPDE) of parabolic type. Such equations arise in many applications of probability theory (see [12] and [23]). For instance, the conditional density in nonlinear filtering problems for a partially observable diffusion process obeys a SPDE and the density of a super-diffusion process also satisfies a SPDE when the dimension of the space domain is 1. If  $d_1 = 3$ , the motion of a random string can be modeled by a stochastic parabolic partial differential system (see [2] and [22]).

General  $L_p$ -theory with  $p \geq 2$  for stochastic parabolic *equations* (not systems) has been well studied. An  $L_p$ -theory of SPDEs with space domain  $\mathbb{R}^n$  was first introduced by Krylov in [12] (cf. [14] for  $L_2$ -theory), and since then the results were extended for SPDEs defined on arbitrary  $C^1$  domains  $\mathcal{O}$  in  $\mathbb{R}^d$  by Krylov, his collaborators and many other mathematicians (see, for instance, [15], [16], [7], [6], [18] and references therein). On the contrary  $L_p$ -theory of general systems of type (1.1) is not available in the literature except  $L_p$ -theory of the system with the Laplace operator (see, for instance, [21], [20] and the reference therein).

Our goal in this article is to prove unique solvability of the systems of type (1.1) in Sobolev spaces with weights. It is known that unless certain compatibility conditions (see, for instance, [1]) are fulfilled, the second and higher derivatives of solutions blow up near the boundary (see [14]). Hence, we measure this blow-up by using appropriate weights. By the way, the Hölder space approach does not allow one to obtain results of reasonable generality (see [16] for details).

We extend the results for single equations in [6], [8], [12], [15], and [16] to the case of the systems under the algebraic condition (2.3) for the leading coefficients  $a_{kr}^{ij}, \sigma_{kr,m}^i$  and very minimal smoothness conditions for the coefficients. Under these assumptions  $a_{kr}^{ij}, \sigma_{kr,m}^i$  are allowed to oscillate to a great extent near the boundary, and  $b_{kr}^i, c_{kr}, \nu_{kr,m}$  may blow up fast near the boundary. For instance, for the case  $d = d_1 = 1$  with the space domain  $\mathbb{R}_+$  we allow  $a := a_{11}^{11}$  to behave like  $2 + \cos |\ln x|^\alpha$  near  $x = 0$ , where  $\alpha \in (0, 1)$  (see Remark 4.7). In this case the oscillation of  $a(t, x)$  increases to infinity as  $x$  approaches the boundary.

For the stability of the numerical solution of (1.1),  $W_2^1$ -theory may be enough in most cases. But, we are interested in the regularity of the solutions and we are aiming at  $W_p^n$  theory. *However*, unlike the results for single equations in [6], [8], [12], and [16], we were able to obtain only  $W_2^n$ -estimates instead of  $W_p^n$ -estimates due to many technical difficulties at this point. For instance, the proofs of Lemma 3.7 and Lemma 3.8 below are not working for  $p > 2$ . Nevertheless, we believe that  $W_2^n$ -theory of the system is a main basis for  $W_p^n$ -theory. The evidences are the results for single equations. For instance, in [9]  $W_p^n$ -theory is established based on Hardy-Littlewood (HL) theorem, Fefferman-Stein (FS) theorem, and  $W_2^n$ -theory. In the future we plan to develop  $W_p^n$ -theory of the system (1.1) by constructing weighted version of HL and FS theorems and using the result in this article.

The organization of this article is as follows. Section 2 handles the Cauchy problem. In section 3 we prove the result with space domain  $\mathbb{R}_+^d$  and in section 4 we finally prove the results on any

bounded  $C^1$ -domains.

In this article  $\mathbb{R}^d$  stands for the Euclidean space of points  $x = (x^1, \dots, x^d)$ ,  $\mathbb{R}_+^d = \{x \in \mathbb{R}^d : x^1 > 0\}$  and  $B_r(x) := \{y \in \mathbb{R}^d : |x - y| < r\}$ . For a function  $u(x)$  we denote

$$u_{x^i} = \frac{\partial u}{\partial x^i} = D_i u, \quad D^\beta u = D_1^{\beta_1} \cdot \dots \cdot D_d^{\beta_d} u, \quad |\beta| = \beta_1 + \dots + \beta_d$$

for the multi-indices  $\beta = (\beta_1, \dots, \beta_d)$ ,  $\beta_i \in \{0, 1, 2, \dots\}$ . By  $c = c(\dots)$  or  $N = N(\dots)$  we mean that the constant  $c$  or  $N$  depends only on what are in parenthesis. Throughout the article, for functions depending on  $\omega, t, x$ , the argument  $\omega \in \Omega$  will be omitted.

## 2 The system with the space domain $\mathcal{O} = \mathbb{R}^d$

In this section we develop a  $W_2^n$ -theory of the Cauchy problem with the system (1.1). For this we don't need weights yet since we don't have a boundary.

Let  $(\Omega, \mathcal{F}, P)$  be a complete probability space and  $\{\mathcal{F}_t : t \geq 0\}$  be a filtration such that  $\mathcal{F}_0$  contains all  $P$ -null sets of  $\Omega$ . By  $\mathcal{P}$  we denote the predictable  $\sigma$ -algebra on  $\Omega \times (0, \infty)$ . Let  $\{w_t^m\}_{m=1}^\infty$  be independent one-dimensional  $\{\mathcal{F}_t\}$ -adapted Wiener processes defined on  $(\Omega, \mathcal{F}, P)$  and  $C_0^\infty := C_0^\infty(\mathbb{R}^d; \mathbb{R}^{d_1})$  denote the set of all  $\mathbb{R}^{d_1}$ -valued infinitely differentiable functions with compact support in  $\mathbb{R}^d$ . By  $\mathcal{D}$  we denote the space of  $\mathbb{R}^d$ -valued distributions on  $C_0^\infty$ ; precisely, for  $u \in \mathcal{D}$  and  $\phi \in C_0^\infty$  we define  $(u, \phi) \in \mathbb{R}^d$  with components  $(u, \phi)^k = (u^k, \phi^k)$ ,  $k = 1, \dots, d_1$ . Here, each  $u^k$  is a usual  $\mathbb{R}$ -valued distribution defined on  $C^\infty(\mathbb{R}^d; \mathbb{R})$ .

We define  $L_p = L_p(\mathbb{R}^d; \mathbb{R}^{d_1})$  as the space of all  $\mathbb{R}^{d_1}$ -valued functions  $u = (u^1, \dots, u^{d_1})$  satisfying

$$\|u\|_{L_p}^p := \sum_{k=1}^{d_1} \|u^k\|_{L_p(\mathbb{R}^d)}^p < \infty.$$

Let  $p \in [2, \infty)$  and  $\gamma \in (-\infty, \infty)$ . We define the space of Bessel potential  $H_p^\gamma = H_p^\gamma(\mathbb{R}^d; \mathbb{R}^{d_1})$  as the space of all distributions  $u$  such that  $(1 - \Delta)^{\gamma/2} u \in L_p$ , where we define each component of it by

$$((1 - \Delta)^{\gamma/2} u)^k = (1 - \Delta)^{\gamma/2} u^k$$

and the operator  $(1 - \Delta)^{\gamma/2}$  is defined by

$$(1 - \Delta)^{\gamma/2} f = \text{the inverse Fourier transform of } (1 + |\xi|^2)^{\gamma/2} \mathcal{F}(f)(\xi)$$

with  $\mathcal{F}(f)$  the Fourier transform of  $f$ . The norm is given by

$$\|u\|_{H_p^\gamma} := \|(1 - \Delta)^{\gamma/2} u\|_{L_p}.$$

Then,  $H_p^\gamma$  equipped with the given norm is a Banach space and  $C_0^\infty$  is dense in  $H_p^\gamma$  (see [24]). For non-negative integer  $\gamma = 0, 1, 2, \dots$ , it turns out that

$$H_p^\gamma = W_p^\gamma := \{u : D^\alpha u \in L_p, \forall \alpha, |\alpha| \leq \gamma\}.$$

It is well known that the first order differentiation operators,  $\partial_i : H_p^\gamma(\mathbb{R}^d; \mathbb{R}) \rightarrow H_p^{\gamma-1}(\mathbb{R}^d; \mathbb{R})$  given by  $u \rightarrow u_{x^i}$  ( $i = 1, 2, \dots, d$ ), are bounded. On the other hand, for  $u \in H_p^\gamma(\mathbb{R}^d; \mathbb{R})$ , if  $\text{supp}(u) \subset (a, b) \times \mathbb{R}^{d-1}$  with  $-\infty < a < b < \infty$ , we have

$$\|u\|_{H_p^\gamma(\mathbb{R}^d; \mathbb{R})} \leq c(d, \gamma, a, b) \|u_x\|_{H_p^{\gamma-1}(\mathbb{R}^d; \mathbb{R})} \quad (2.1)$$

(see, for instance, Remark 1.13 in [13]).

By  $\ell_2$  we denote the set of all real-valued sequences  $e = (e_1, e_2, \dots)$  with the inner product  $(e, f)_{\ell_2} = \sum_{m=1}^\infty e_m f_m$  and the norm  $|e|_{\ell_2} := (e, e)_{\ell_2}^{1/2}$ . If  $g = (g^1, g^2, \dots, g^{d_1})$  and each  $g^k$  is an  $\ell_2$ -valued function, then we define

$$\|g\|_{H_p^\gamma(\ell_2)}^p := \sum_{k=1}^{d_1} \|(1 - \Delta)^{\gamma/2} g^k\|_{\ell_2}^p \|g^k\|_{L_p}^p.$$

For a fixed time  $T < \infty$ , we define the stochastic Banach spaces

$$\mathbb{H}_p^\gamma(T) = \mathbb{H}_p^\gamma(\mathbb{R}^d, T) := L_p(\Omega \times (0, T], \mathcal{P}, H_p^\gamma), \quad \mathbb{H}_p^\gamma(T, \ell_2) := L_p(\Omega \times (0, T], \mathcal{P}, H_p^\gamma(\ell_2)),$$

$$\mathbb{L}_p(T) := \mathbb{H}_p^0(T), \quad \mathbb{L}_p(T, \ell_2) = \mathbb{H}_p^0(T, \ell_2)$$

with the norms given by

$$\|u\|_{\mathbb{H}_p^\gamma(T)}^p = \mathbb{E} \int_0^T \|u(t)\|_{H_p^\gamma}^p dt, \quad \|g\|_{\mathbb{H}_p^\gamma(T, \ell_2)}^p = \mathbb{E} \int_0^T \|g(t)\|_{H_p^\gamma(\ell_2)}^p dt.$$

Finally, we set  $U_p^\gamma := L_p(\Omega, \mathcal{F}_0, H_p^{\gamma-2/p})$  for the initial data of the Cauchy problem. The Banach space  $\mathcal{H}_p^{\gamma+2}(T)$  below is modified from  $\mathbb{R}$ -valued version in [12] to the  $\mathbb{R}^{d_1}$ -valued version.

**Definition 2.1.** For a  $\mathcal{D}$ -valued function  $u = (u^1, \dots, u^{d_1}) \in \mathbb{H}_p^{\gamma+2}(T)$ , we write  $u \in \mathcal{H}_p^{\gamma+2}(T)$  if  $u(0, \cdot) \in U_p^{\gamma+2}$ , and there exist  $f \in \mathbb{H}_p^\gamma(T)$ ,  $g \in \mathbb{H}_p^{\gamma+1}(T, \ell_2)$  such that, for any  $\phi \in C_0^\infty$ , (a.s.) the equality

$$(u^k(t, \cdot), \phi) = (u^k(0, \cdot), \phi) + \int_0^t (f^k(s, \cdot), \phi) ds + \sum_{m=1}^\infty \int_0^t (g_m^k(s, \cdot), \phi) dw_s^m \quad (2.2)$$

holds for each  $k = 1, \dots, d_1$  and  $t \in (0, T]$ . The norm of  $u$  in  $\mathcal{H}_p^{\gamma+2}(T)$  is defined by

$$\|u\|_{\mathcal{H}_p^{\gamma+2}(T)} = \|u\|_{\mathbb{H}_p^{\gamma+2}(T)} + \|f\|_{\mathbb{H}_p^\gamma(T)} + \|g\|_{\mathbb{H}_p^{\gamma+1}(T, \ell_2)} + \|u(0, \cdot)\|_{U_p^{\gamma+2}}.$$

We write (2.2) in the following simplified ways,

$$u(t) = u(0) + \int_0^t f(s) ds + \int_0^t g_m(s) dw_s^m \quad \text{or} \quad du = f dt + g_m dw_t^m, \quad t \in (0, T]$$

and we say that  $du = f dt + g_m dw_t^m$  holds in the sense of distributions.

For any  $m \times n$  real-valued matrix  $C = (c_{kr})$ , we define its norm by

$$|C| := \sqrt{\sum_{k=1}^m \sum_{r=1}^n (c_{kr})^2}.$$

We set  $A^{ij} = (a_{kr}^{ij})$ ,  $\Sigma^i = (\sigma_{kr}^i)$ , and  $\mathcal{A}^{ij} = (\alpha_{kr}^{ij})$ , where

$$\alpha_{kr}^{ij} = \frac{1}{2} \sum_{l=1}^{d_1} (\sigma_{lk}^i, \sigma_{lr}^j)_{\ell_2}, \quad \sigma_{kr}^i = (\sigma_{kr,1}^i, \sigma_{kr,2}^i, \dots).$$

Throughout the article we assume the followings.

**Assumption 2.2.** (i) The coefficients  $a_{kr}^{ij}, b_{kr}^i, c_{kr}, \sigma_{kr,m}^i$ , and  $\nu_{kr,m}$  are  $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable.  
(ii) There exist finite constants  $\delta, K^j, L > 0$  so that

$$\delta |\xi|^2 \leq \xi_i^* (A^{ij} - \mathcal{A}^{ij}) \xi_j, \quad (2.3)$$

$$|A^{1j}| \leq K^j, \quad |\mathcal{A}^{ij}| \leq L, \quad i, j = 1, 2, \dots, d \quad (2.4)$$

hold for any  $\omega \in \Omega$ ,  $t \geq 0$ ,  $x \in \mathbb{R}^d$ , where  $\xi$  is any (real)  $d_1 \times d$  matrix,  $\xi_i$  is the  $i$ th column of  $\xi$ ,  $*$  denotes the matrix transpose, and again the summations on  $i, j$  are understood.

Before we consider the general system (1.1), we give a  $W_2^n$ -theory for the Cauchy problem with the coefficients independent of  $x$ :

$$du^k = (a_{kr}^{ij} u_{x^i x^j}^r + f^k) dt + (\sigma_{kr,m}^i u_{x^i}^r + g_m^k) dw_t^m, \quad u^k(0, \cdot) = u_0^k(\cdot), \quad (2.5)$$

where  $i, j = 1, 2, \dots, d$ ,  $k, r = 1, 2, \dots, d_1$ ,  $m = 1, 2, \dots$ ; recall that we are using summation notation on  $i, j, r$ .

**Theorem 2.3.** Let  $a_{kr}^{ij} = a_{kr}^{ij}(\omega, t)$  and  $\sigma_{kr,m}^i = \sigma_{kr,m}^i(\omega, t)$ . Then for any  $f \in \mathbb{H}_2^\gamma(T)$ ,  $g \in \mathbb{H}_2^{\gamma+1}(T, \ell_2)$ , and  $u_0 \in U_2^{\gamma+2}$ , the problem (2.5) has a unique solution  $u \in \mathcal{H}_2^{\gamma+2}(T)$  and for this solution we have

$$\|u_{xx}\|_{\mathbb{H}_2^\gamma(T)} \leq c \left( \|f\|_{\mathbb{H}_2^\gamma(T)} + \|g\|_{\mathbb{H}_2^{\gamma+1}(T, \ell_2)} + \|u_0\|_{U_2^{\gamma+2}} \right), \quad (2.6)$$

$$\|u\|_{\mathbb{H}_2^{\gamma+2}(T)} \leq c e^{cT} \left( \|f\|_{\mathbb{H}_2^\gamma(T)} + \|g\|_{\mathbb{H}_2^{\gamma+1}(T, \ell_2)} + \|u_0\|_{U_2^{\gamma+2}} \right), \quad (2.7)$$

where  $c = c(d, d_1, \gamma, \delta, K^j, L)$ .

*Proof.* Let  $\Delta$  denote the usual Laplace operator. By Theorem 4.10 and Theorem 5.1 in [12], for each  $k$ , the single equation

$$du^k = (\delta \Delta u^k + f^k) dt + g_m^k dw_t^m, \quad u^k(0) = u_0^k$$

has a solution  $u^k \in \mathcal{H}_2^{\gamma+2}(T)$ . For  $\lambda \in [0, 1]$  and  $d_1 \times d_1$  identity matrix  $I$  we define

$$\begin{aligned} \bar{A}_\lambda^{ij} &= (\bar{a}_{kr,\lambda}^{ij}) := (1 - \lambda) (A^{ij} - \mathcal{A}^{ij}) + \delta^{ij} \lambda \delta I \\ &= ((1 - \lambda) A^{ij} + \delta^{ij} \lambda \delta I) - (1 - \lambda) \mathcal{A}^{ij} = A_\lambda^{ij} - \mathcal{A}_\lambda^{ij}, \end{aligned}$$

where  $A_\lambda^{ij} := (1 - \lambda) A^{ij} + \delta^{ij} \lambda \delta I$ ,  $\mathcal{A}_\lambda^{ij} := (1 - \lambda) \mathcal{A}^{ij}$ . Then

$$|A_\lambda^{ij}| \leq |A^{ij}|, \quad |\mathcal{A}_\lambda^{ij}| \leq |\mathcal{A}^{ij}|, \quad \delta |\xi|^2 \leq \sum_{i,j} \xi_i^* \bar{A}_\lambda^{ij} \xi_j$$

for any  $d_1 \times d$ -matrix  $\xi$ . Thus, having the method of continuity in mind (see the proof of Theorem 5.1 in [12] for the details), we only prove that the a priori estimates (2.7) and (2.6) hold given that a solution  $u$  already exists.

**Step 1.** Assume  $\gamma = 0$ . Applying the stochastic product rule  $d|u^k|^2 = 2u^k du^k + du^k du^k$  for each  $k$ , we have

$$\begin{aligned} |u^k(t)|^2 &= |u_0^k|^2 + \int_0^t \left[ 2u^k(a_{kr}^{ij}u_{x^i x^j}^r + f^k) + |\sigma_{kr}^i u_{x^i}^r + g^k|_{\ell_2}^2 \right] ds \\ &\quad + \int_0^t 2u^k(\sigma_{kr,m}^i u_{x^i}^r + g_m^k) dw_s^m, \quad t > 0. \end{aligned} \quad (2.8)$$

Making the summation on  $r, i$  appeared, we note that

$$\begin{aligned} \sum_k \left| \sum_{r,i} \sigma_{kr}^i u_{x^i}^r + g^k \right|_{\ell_2}^2 &= \sum_k \left[ \left| \sum_{r,i} \sigma_{kr}^i u_{x^i}^r \right|_{\ell_2}^2 + 2(\sum_{r,i} \sigma_{kr}^i u_{x^i}^r, g^k)_{\ell_2} + |g^k|_{\ell_2}^2 \right] \\ &= 2 \sum_{i,j} (u_{x^i})^* \mathcal{A}^{ij} u_{x^j} + 2 \sum_{k,r,i} (\sigma_{kr}^i u_{x^i}^r, g^k)_{\ell_2} + \sum_k |g^k|_{\ell_2}^2. \end{aligned}$$

By taking expectation, integrating with respect to  $x$ , and using integrating by parts in order, we get from (2.8)

$$\begin{aligned} &\mathbb{E} \int_{\mathbb{R}^d} |u(t)|^2 dx + 2 \mathbb{E} \int_0^t \int_{\mathbb{R}^d} \sum_{i,j} (u_{x^i})^* (A^{ij} - \mathcal{A}^{ij}) u_{x^j} dx ds \\ &= \mathbb{E} \int_{\mathbb{R}^d} |u_0|^2 dx + \mathbb{E} \int_0^t \int_{\mathbb{R}^d} \left[ 2u^* f + 2 \sum_{k,r,i} (\sigma_{kr}^i u_{x^i}^r, g^k)_{\ell_2} + \sum_k |g^k|_{\ell_2}^2 \right] dx ds. \end{aligned} \quad (2.9)$$

Note that

$$\begin{aligned} 2 \left| \sum_{k,r,i} (\sigma_{kr}^i u_{x^i}^r, g^k)_{\ell_2} \right| &\leq 2 \sum_k \left| \sum_{r,i} \sigma_{kr}^i u_{x^i}^r \right|_{\ell_2} |g^k|_{\ell_2} \\ &\leq \sum_k \left( \frac{\varepsilon}{2} \left| \sum_{r,i} \sigma_{kr}^i u_{x^i}^r \right|_{\ell_2}^2 + \frac{2}{\varepsilon} |g^k|_{\ell_2}^2 \right) \\ &\leq \frac{\varepsilon}{2} |u_x|^2 \sum_{k,r,i} |\sigma_{kr}^i|_{\ell_2}^2 + \frac{2}{\varepsilon} \sum_k |g^k|_{\ell_2}^2 \\ &= \varepsilon |u_x|^2 \sum_{r,i} |\alpha_{rr}^{ii}|^2 + \frac{2}{\varepsilon} \sum_k |g^k|_{\ell_2}^2 \end{aligned} \quad (2.10)$$

for any  $\varepsilon > 0$ . Hence, it follows that

$$\begin{aligned} &\mathbb{E} \int_{\mathbb{R}^d} |u(t)|^2 dx + 2\delta \mathbb{E} \int_0^t \int_{\mathbb{R}^d} |u_x|^2 dx ds \\ &\leq \mathbb{E} \int_{\mathbb{R}^d} |u_0|^2 dx + \varepsilon \cdot d \cdot L^2 \mathbb{E} \int_0^t \int_{\mathbb{R}^d} |u_x|^2 dx ds + \mathbb{E} \int_0^t \int_{\mathbb{R}^d} |u|^2 dx ds \\ &\quad + \mathbb{E} \int_0^t \int_{\mathbb{R}^d} |f|^2 dx ds + c \mathbb{E} \sum_k \int_0^t \int_{\mathbb{R}^d} |g^k|_{\ell_2}^2 dx ds. \end{aligned} \quad (2.11)$$

Similarly, for  $v = u_{x^n}$  with any  $n = 1, 2, \dots, d$ , we get (see (2.9))

$$\begin{aligned}
& \mathbb{E} \int_{\mathbb{R}^d} |v(t)|^2 dx + 2\delta \mathbb{E} \int_0^t \int_{\mathbb{R}^d} |v_x|^2 dx ds \\
&= \mathbb{E} \int_{\mathbb{R}^d} |(u_0)_{x^n}|^2 dx + \mathbb{E} \int_0^t \int_{\mathbb{R}^d} \left[ -2v_{x^n}^* f + 2 \sum_{k,r,i} (\sigma_{kr}^i v_{x^i}^r, g_{x^n}^k)_{\ell_2} + \sum_k |g_{x^n}^k|_{\ell_2}^2 \right] dx ds. \\
&\leq \|u_0\|_{U_2^2}^2 + \varepsilon \|u_{xx}\|_{\mathbb{L}_2(t)}^2 + c \|f\|_{\mathbb{L}_2(t)}^2 + c \|g_x\|_{\mathbb{L}_2(t, \ell_2)}^2.
\end{aligned} \tag{2.12}$$

Choosing small  $\varepsilon$  and considering all  $n$ , we have (2.6). Now, (2.12), (2.11) and Gronwall's inequality easily lead to (2.7).

**Step 2.** Let  $\gamma \neq 0$ . The result of this case easily follows from the fact that  $(1 - \Delta)^{\mu/2} : H_p^\gamma \rightarrow H_p^{\gamma-\mu}$  is an isometry for any  $\gamma, \mu \in \mathbb{R}$  when  $p \in (1, \infty)$ ; indeed,  $u \in \mathcal{H}_2^{\gamma+2}(T)$  is a solution of (2.5) if and only if  $v := (1 - \Delta)^{\gamma/2} u \in \mathcal{H}_2^2(T)$  is a solution of (2.5) with  $(1 - \Delta)^{\gamma/2} f, (1 - \Delta)^{\gamma/2} g, (1 - \Delta)^{\gamma/2} u_0$  in places of  $f, g, u_0$  respectively. Moreover, for instance, we have

$$\begin{aligned}
\|u\|_{\mathbb{H}_2^{\gamma+2}(T)} &= \|v\|_{\mathbb{H}_2^2(T)} \leq ce^{cT} \left( \|(1 - \Delta)^{\gamma/2} f\|_{\mathbb{L}_2(T)} + \|(1 - \Delta)^{\gamma/2} g\|_{\mathbb{H}_2^1(T, \ell_2)} + \|(1 - \Delta)^{\gamma/2} u_0\|_{U_2^2} \right) \\
&= ce^{cT} \left( \|f\|_{\mathbb{H}_2^\gamma(T)} + \|g\|_{\mathbb{H}_2^{\gamma+1}(T, \ell_2)} + \|u_0\|_{U_2^{\gamma+2}} \right).
\end{aligned}$$

The theorem is proved.  $\square$

Now we extend Theorem 2.3 to the case of the Cauchy problem with variable coefficients. Fix  $\varepsilon_0 > 0$ . For  $\gamma \in \mathbb{R}$  let us define  $|\gamma|_+ = |\gamma|$  if  $|\gamma| = 0, 1, 2, \dots$  and  $|\gamma|_+ = |\gamma| + \varepsilon_0$  otherwise. Then we define

$$B^{|\gamma|_+} = \begin{cases} B(\mathbb{R}^d) & : \gamma = 0 \\ C^{|\gamma|-1,1}(\mathbb{R}^d) & : |\gamma| = 1, 2, \dots \\ C^{|\gamma|+\kappa}(\mathbb{R}^d) & : \text{otherwise,} \end{cases}$$

where  $B$  is the space of bounded functions, and  $C^{|\gamma|-1,1}$  and  $C^{|\gamma|+\kappa}$  are the usual Hölder spaces. The Banach space  $B^{|\gamma|_+}$  is also defined for  $\ell_2$ -valued functions. For instance, if  $g = (g_1, g_2, \dots)$ , then  $|g|_{B^0} = \sup_x |g(x)|_{\ell_2}$  and

$$|g|_{C^{n-1,1}} = \sum_{|\alpha| \leq n-1} |D^\alpha g|_{B^0} + \sum_{|\alpha|=n-1} \sup_{x \neq y} \frac{|D^\alpha g(x) - D^\alpha g(y)|_{\ell_2}}{|x - y|}.$$

Here is the main result of this section.

**Theorem 2.4.** Assume that the coefficients  $a_{kr}^{ij}, \sigma_{kr}^i$  are uniformly continuous in  $x$ , that is, for any  $\varepsilon > 0$  there exists  $\delta = \delta(\varepsilon) > 0$  so that for any  $\omega, t > 0, i, j, k, r$ ,

$$|a_{kr}^{ij}(\omega, t, x) - a_{kr}^{ij}(\omega, t, y)| + |\sigma_{kr}^i(\omega, t, x) - \sigma_{kr}^i(\omega, t, y)|_{\ell_2} < \varepsilon, \quad \text{if } |x - y| < \delta.$$

Also, assume for any  $\omega, t > 0, i, j, k, r$ ,

$$|a_{kr}^{ij}(\omega, t, \cdot)|_{|\gamma|_+} + |b_{kr}^i(\omega, t, \cdot)|_{|\gamma|_+} + |c_{kr}(\omega, t, \cdot)|_{|\gamma|_+} + |\sigma_{kr}^i(\omega, t, \cdot)|_{|\gamma+1|_+} + |\nu_{kr}(\omega, t, \cdot)|_{|\gamma+1|_+} < L.$$

Then for any  $f \in \mathbb{H}_2^\gamma(T)$ ,  $g \in \mathbb{H}_2^{\gamma+1}(T, \ell_2)$  and  $u_0 \in U_2^{\gamma+2}$ , the Cauchy problem (1.1) has a unique solution  $u \in \mathcal{H}_2^{\gamma+2}(T)$ , and for this solution we have

$$\|u\|_{\mathbb{H}_2^{\gamma+2}(T)} \leq c \left( \|f\|_{\mathbb{H}_2^\gamma(T)} + \|g\|_{\mathbb{H}_2^{\gamma+1}(T, \ell_2)} + \|u_0\|_{U_2^{\gamma+2}} \right),$$

where  $c = c(d, d_1, \gamma, \delta, K^j, L, T)$ .

*Proof.* It is enough to repeat the proof of Theorem 5.2 in [12], where the theorem is proved for single equations. The only difference is that one needs to use Theorem 2.3 of this article, instead of Theorem 4.10 in [12]. We leave the details to the reader.  $\square$

### 3 The system with the space domain $\mathcal{O} = \mathbb{R}_+^d$

In this section we study a  $W_2^n$ -theory of the initial value problem with the space domain  $\mathbb{R}_+^d$ . We use the Banach spaces introduced in [13]. Let  $\zeta \in C_0^\infty(\mathbb{R}_+)$  be a function satisfying

$$\sum_{n=-\infty}^{\infty} \zeta(e^{n+x}) > c > 0, \quad \forall x \in \mathbb{R}, \quad (3.1)$$

where  $c$  is a constant. It is each to check that any nonnegative function  $\zeta$  with the property  $\zeta > 0$  on  $[1, e]$  satisfies (3.1). For  $\theta, \gamma \in \mathbb{R}$ , let  $H_{p, \theta}^\gamma$  be the set of all distributions  $u = (u^1, u^2, \dots, u^{d_1})$  on  $\mathbb{R}_+^d$  such that

$$\|u\|_{H_{p, \theta}^\gamma}^p := \sum_{n \in \mathbb{Z}} e^{n\theta} \|\zeta(\cdot) u(e^n \cdot)\|_{H_p^\gamma}^p < \infty. \quad (3.2)$$

If  $g = (g^1, g^2, \dots, g^{d_1})$  and each  $g^k$  is an  $\ell_2$ -valued function, then we define

$$\|g\|_{H_{p, \theta}^\gamma(\ell_2)}^p = \sum_{n \in \mathbb{Z}} e^{n\theta} \|\zeta(\cdot) g(e^n \cdot)\|_{H_p^\gamma(\ell_2)}^p.$$

It is known ([13]) that up to equivalent norms the space  $H_{p, \theta}^\gamma$  is independent of the choice of  $\zeta$ . Also, for any  $\eta \in C_0^\infty(\mathbb{R}_+)$ , we have

$$\sum_{n=-\infty}^{\infty} e^{n\theta} \|u(e^n \cdot) \eta\|_{H_p^\gamma}^p \leq c \sum_{n=-\infty}^{\infty} e^{n\theta} \|u(e^n \cdot) \zeta\|_{H_p^\gamma}^p, \quad (3.3)$$

where  $c$  depends only on  $d, \gamma, \theta, p, \eta, \zeta$ . Furthermore, if  $\gamma$  is a nonnegative integer, then

$$H_{p, \theta}^\gamma = \{u : u, x^1 Du, \dots, (x^1)^{|\alpha|} D^\alpha u \in L_p(\mathbb{R}_+^d, (x^1)^{\theta-d} dx), |\alpha| \leq \gamma\},$$

$$\|u\|_{H_{p, \theta}^\gamma}^p \sim \sum_{|\alpha| \leq \gamma} \int_{\mathbb{R}_+^d} |(x^1)^{|\alpha|} D^\alpha u(x)|^p (x^1)^{\theta-d} dx.$$

Below we collect some other properties of spaces  $H_{p, \theta}^\gamma$ . For  $\mu \in \mathbb{R}$  let  $M^\mu$  be the operator of multiplying by  $(x^1)^\mu$  and  $M = M^1$ .



**Lemma 3.1.** ([13]) (i) Assume that  $\gamma - d/p = m + \nu$  for some  $m = 0, 1, \dots$  and  $\nu \in (0, 1]$ . Then for any  $u \in H_{p,\theta}^\gamma$  and  $i \in \{0, 1, \dots, m\}$ , we have

$$|M^{i+\theta/p} D^i u|_C + [M^{m+\nu+\theta/p} D^m u]_{C^\nu} \leq c \|u\|_{H_{p,\theta}^\gamma}.$$

(ii) Let  $\mu \in \mathbb{R}$ . Then  $M^\mu H_{p,\theta+\mu p}^\gamma = H_{p,\theta}^\gamma$ ,

$$\|u\|_{H_{p,\theta}^\gamma} \leq c \|M^{-\mu} u\|_{H_{p,\theta+\mu p}^\gamma} \leq c \|u\|_{H_{p,\theta}^\gamma}.$$

(iii)  $MD, DM : H_{p,\theta}^\gamma \rightarrow H_{p,\theta}^{\gamma-1}$  are bounded linear operators, and it holds that

$$\|u\|_{H_{p,\theta}^\gamma} \leq c \|u\|_{H_{p,\theta}^{\gamma-1}} + c \|MDu\|_{H_{p,\theta}^{\gamma-1}} \leq c \|u\|_{H_{p,\theta}^\gamma},$$

$$\|u\|_{H_{p,\theta}^\gamma} \leq c \|u\|_{H_{p,\theta}^{\gamma-1}} + c \|DMu\|_{H_{p,\theta}^{\gamma-1}} \leq c \|u\|_{H_{p,\theta}^\gamma}.$$

(iv) The operator  $\mathcal{L} := M^2 \Delta + 2MD_1$  is a bounded operator from  $H_{p,\theta}^\gamma$  onto  $H_{p,\theta}^{\gamma-2}$  with the bounded inverse  $\mathcal{L}^{-1}$  for any  $\gamma$ .

Let us denote

$$\mathbb{H}_{p,\theta}^\gamma(T) = L_p(\Omega \times [0, T], \mathcal{P}, H_{p,\theta}^\gamma), \quad \mathbb{H}_{p,\theta}^\gamma(T, \ell_2) = L_p(\Omega \times [0, T], \mathcal{P}, H_{p,\theta}^\gamma(\ell_2)),$$

$$U_{p,\theta}^\gamma = M^{1-2/p} L_p(\Omega, \mathcal{F}_0, H_{p,\theta}^{\gamma-2/p}), \quad \mathbb{L}_{p,\theta}(T) = \mathbb{H}_{p,\theta}^0(T).$$

The Banach space  $\mathfrak{H}_p^{\gamma+2}(T)$  below is modified from  $\mathbb{R}$ -valued version in [16] to the  $\mathbb{R}^{d_1}$ -valued version.

**Definition 3.2.** We write  $u \in \mathfrak{H}_p^{\gamma+2}(T)$  if  $u = (u^1, \dots, u^{d_1}) \in M\mathbb{H}_{p,\theta}^{\gamma+2}(T)$ ,  $u(0, \cdot) \in U_{p,\theta}^{\gamma+2}$ , and for some  $f \in M^{-1}\mathbb{H}_{p,\theta}^\gamma(T)$ ,  $g \in \mathbb{H}_{p,\theta}^{\gamma+1}(T, \ell_2)$ ,

$$du = f dt + g_m dw_t^m, \quad t \in [0, T]$$

in the sense of distributions. We define the norm by

$$\|u\|_{\mathfrak{H}_p^{\gamma+2}(T)} =: \|M^{-1}u\|_{\mathbb{H}_{p,\theta}^{\gamma+2}(T)} + \|Mf\|_{\mathbb{H}_{p,\theta}^\gamma(T)} + \|g\|_{\mathbb{H}_{p,\theta}^{\gamma+1}(T, \ell_2)} + \|u(0, \cdot)\|_{U_{p,\theta}^{\gamma+2}}. \quad (3.4)$$

**Definition 3.3.** Let  $A^{ij} = (a_{kr}^{ij})$  and  $\Sigma^i = (\sigma_{kr}^i)$  be **independent** of  $x$ . We say that  $(A^{ij}, \Sigma^i, \theta)$  is admissible (with constant  $N$ ) if whenever  $u \in M\mathbb{H}_{2,\theta}^1(T)$  is a solution of the problem

$$\begin{aligned} du^k &= (a_{kr}^{ij} u_{x^i x^j}^r + f^k) dt + (\sigma_{kr,m}^i u_{x^i}^r + g_m^k) dw_t^m, \quad t > 0, \quad x \in \mathbb{R}^d, \\ u^k(0, \cdot) &= u_0^k(\cdot), \end{aligned} \quad (3.5)$$

satisfying  $u \in L_2(\Omega, C([0, T], C_0^2((1/n, n) \times \{x' : |x'| < n\})))$  for some constant  $n > 0$ , it holds that

$$\|M^{-1}u\|_{\mathbb{L}_{2,\theta}(T)}^2 \leq N \left( \|Mf\|_{\mathbb{L}_{2,\theta}(T)}^2 + \|g\|_{\mathbb{L}_{2,\theta}(T, \ell_2)}^2 + \|u_0\|_{U_{2,\theta}^1}^2 \right). \quad (3.6)$$

In Theorem 3.4 below we give some sufficient conditions under which  $(A^{ij}, \Sigma^i, \theta)$  is admissible. We define the symmetric part  $(S^{ij})$  and the diagonal part  $(S_d^{ij})$  of  $A^{ij}$  as follows:

$$S^{ij} = (s_{kr}^{ij}) := (A^{ij} + (A^{ij})^*)/2, \quad S_d^{ij} = (s_{d,kr}^{ij}) := (\delta_{kr} a_{kr}^{ij}) = (\delta_{kr} s_{kr}^{ij}).$$

We also define

$$H^{ij} := A^{ij} - (A^{ij})^*, \quad S_o^{ij} = S^{ij} - S_d^{ij}.$$

Assume that there exist constants  $\alpha, \beta_1, \dots, \beta_d \in [0, \infty)$  such that

$$|H^{1j}| \leq \beta^j \quad \forall j = 1, 2, \dots, d, \quad |S_o^{11}| \leq \alpha. \quad (3.7)$$

We denote

$$K := \sqrt{\sum_{j=1}^d (K^j)^2}, \quad \beta := \sqrt{\sum_{j=1}^d (\beta^j)^2}.$$

**Theorem 3.4.** *Let one of the following four conditions be satisfied:*

$$\theta \in \left( d - \frac{\delta}{2K - \delta}, d + \frac{\delta}{2K + \delta} \right), \quad (3.8)$$

$$\theta \in [d, d+1), \quad 8(d+1-\theta)\delta^2 - (\theta-d)\beta^2 > 0, \quad (3.9)$$

$$\theta \in (d-1, d], \quad 2\delta(d+1-\theta)^2 - 2(d+1-\theta)(d-\theta)\beta - 4(d-\theta)(d+1-\theta)K^1 > 0, \quad (3.10)$$

$$\theta \in (d-1, d], \quad \left[ \frac{d-\theta}{d+1-\theta}(\beta + 2\alpha) + \varepsilon \right] |\xi|^2 \leq \xi_i^* \left( A^{ij} - \mathcal{A}^{ij} - 2\frac{d-\theta}{d+1-\theta} S_d^{ij} \right) \xi_j, \quad (3.11)$$

where  $\varepsilon > 0$ ,  $\xi$  is any (real)  $d_1 \times d$  matrix and  $\xi_i$  is the  $i$ th column of  $\xi$ . Then there exists a constant  $N = N(\theta, \delta, K) > 0$  so that  $(A^{ij}, \Sigma^i, \theta)$  is admissible with constant  $N$ .

*Remark 3.5.* (i) If  $A^{1j}$  are symmetric, i.e.,  $\beta = 0$ , then (3.10) combined with (3.9) is the same as the condition  $\theta \in (d - \frac{\delta}{2K - \delta}, d + 1)$ , which is weaker than (3.8).

(ii) If  $A^{ij}$  are diagonal matrices and  $\Sigma^i = 0$ , then  $\alpha = \beta^i = 0$  and  $A^{ij} = S_d^{ij}$ . Since  $1 - 2(d-\theta)/(d+1-\theta) > 0$  for  $\theta > d-1$ , (3.11) combined with (3.9) is the same as the condition  $\theta \in (d-1, d+1)$ . This is the case when the equations in the system is not correlated.

*Remark 3.6.* We do not know how sharp the above conditions are. However, it is known ([13]) that if  $\theta \notin (d-1, d+1)$ , then Theorem 3.4 is false even for the (deterministic) heat equation  $u_t = \Delta u + f$ . i.e.,  $(\delta^{ij}I, 0, \theta)$  is not admissible for such  $\theta$ .

Theorem 3.4 is proved in the following two lemmas.

**Lemma 3.7.** *Assume that  $a_{kr}^{ij}, \sigma_{kr,m}^i$  are independent of  $x$ , and*

$$\theta \in \left( d - \frac{\delta}{2K - \delta}, d + \frac{\delta}{2K + \delta} \right). \quad (3.12)$$

Let  $u \in M\mathbb{H}_{2,\theta}^1(T)$  be a solution of (3.5) so that  $u \in L_2(\Omega, C([0, T], C_0^2((1/n, n) \times \{x' : |x'| < n\})))$  for some  $n > 0$ . Then we have

$$\|M^{-1}u\|_{\mathbb{L}_{2,\theta}(T)}^2 \leq N \left( \|Mf\|_{\mathbb{L}_{2,\theta}(T)}^2 + \|g\|_{\mathbb{L}_{2,\theta}(T,\ell_2)}^2 + \|u_0\|_{U_{2,\theta}^1}^2 \right), \quad (3.13)$$

where  $N = N(d, d_1, \delta, \theta, K, L)$ .

*Proof.* As in the proof of Theorem 2.3, applying the stochastic product rule  $d|u^k|^2 = 2u^k du^k + du^k du^k$  for each  $k$ , we have

$$\begin{aligned} |u^k(t)|^2 &= |u_0^k|^2 + \int_0^t \left[ 2u^k(a_{kr}^{ij}u_{x^i x^j}^r + f^k) + |\sigma_{kr}^i u_{x^i}^r + g^k|_{\ell_2}^2 \right] ds \\ &\quad + \int_0^t 2u^k(\sigma_{kr,m}^i u_{x^i}^r + g_m^k) dw_s^m, \quad t > 0, \end{aligned}$$

where the summations on  $i, j, r$  are understood. Denote  $c := \theta - d$ . For each  $k$ , we have

$$\begin{aligned} 0 &\leq \mathbb{E} \int_{\mathbb{R}_+^d} |u^k(T, x)|^2 (x^1)^c dx \\ &= \mathbb{E} \int_{\mathbb{R}_+^d} |u^k(0, x)|^2 (x^1)^c dx + 2\mathbb{E} \int_0^T \int_{\mathbb{R}_+^d} a_{kr}^{ij} u^k u_{x^i x^j}^r (x^1)^c dx ds \\ &\quad + \mathbb{E} \int_0^T \int_{\mathbb{R}_+^d} |\sigma_{kr}^i u_{x^i}^r|_{\ell_2}^2 (x^1)^c dx ds + 2\mathbb{E} \int_0^T \int_{\mathbb{R}_+^d} (M^{-1}u^k)(Mf^k)(x^1)^c dx ds \\ &\quad + 2\mathbb{E} \int_0^T \int_{\mathbb{R}_+^d} (\sigma_{kr}^i u_{x^i}^r, g^k)_{\ell_2} (x^1)^c dx ds + \mathbb{E} \int_0^T \int_{\mathbb{R}_+^d} |g^k|_{\ell_2}^2 (x^1)^c dx ds. \end{aligned} \quad (3.14)$$

Note that, by integration by parts, the second term in the right hand side of (3.14) is

$$-2\mathbb{E} \int_0^T \int_{\mathbb{R}_+^d} a_{kr}^{ij} u_{x^i}^k u_{x^j}^r (x^1)^c dx ds - 2c\mathbb{E} \int_0^T \int_{\mathbb{R}_+^d} (a_{kr}^{1j} u_{x^j}^r)(M^{-1}u^k)(x^1)^c dx ds. \quad (3.15)$$

By summing up the terms in the right hand side of (3.14) over  $k$  and rearranging the terms, we get

$$\begin{aligned} &2\mathbb{E} \int_0^T \int_{\mathbb{R}_+^d} u_{x^i}^* (A^{ij} - \mathcal{A}^{ij}) u_{x^j} (x^1)^c dx ds \\ &\leq -2c\mathbb{E} \int_0^T \int_{\mathbb{R}_+^d} a_{kr}^{1j} u_{x^j}^r (M^{-1}u^k)(x^1)^c dx ds + \varepsilon \left( \|M^{-1}u\|_{\mathbb{L}_{2,\theta}(T)}^2 + \|u_x\|_{\mathbb{L}_{2,\theta}(T)}^2 \right) \\ &\quad + c(\varepsilon) \left( \|Mf\|_{\mathbb{L}_{2,\theta}(T)}^2 + \|g\|_{\mathbb{L}_{2,\theta}(T)}^2 \right) + \|u(0)\|_{U_{2,\theta}^1}^2 \\ &\leq |c| \left( \kappa \|u_x\|_{\mathbb{L}_{2,\theta}(T)}^2 + K^2 \kappa^{-1} \|M^{-1}u\|_{\mathbb{L}_{2,\theta}(T)}^2 \right) + \varepsilon \left( \|M^{-1}u\|_{\mathbb{L}_{2,\theta}(T)}^2 + \|u_x\|_{\mathbb{L}_{2,\theta}(T)}^2 \right) \\ &\quad + c(\varepsilon) \left( \|Mf\|_{\mathbb{L}_{2,\theta}(T)}^2 + \|g\|_{\mathbb{L}_{2,\theta}(T)}^2 \right) + \|u(0)\|_{U_{2,\theta}^1}^2, \end{aligned} \quad (3.16)$$

where for the second inequality we used (2.4), (2.10), and the fact: for any vectors  $v, w \in \mathbb{R}^n$  and  $\kappa > 0$ ,

$$|\langle A^{1j}v, w \rangle| \leq |A^{1j}v||w| \leq K^j|v||w| \leq \frac{1}{2}(\kappa|v|^2 + \kappa^{-1}(K^j)^2|w|^2);$$

$\kappa, \varepsilon$  will be decided below. Condition (2.3), inequality (3.16) and the inequality

$$\|M^{-1}u\|_{L_{2,\theta}}^2 \leq \frac{4}{(d+1-\theta)^2} \|u_x\|_{L_{2,\theta}}^2 \quad (3.17)$$

(see Corollary 6.2 in [13]) lead us to

$$\begin{aligned} &2\delta \|u_x\|_{\mathbb{L}_{2,\theta}(T)}^2 - |c| \left( \kappa + \frac{4K^2}{\kappa(d+1-\theta)^2} \right) \|u_x\|_{\mathbb{L}_{2,\theta}(T)}^2 \\ &\leq \varepsilon \left( \frac{4}{(d+1-\theta)^2} + d^2 L \right) \|u_x\|_{\mathbb{L}_{2,\theta}(T)}^2 + c(\varepsilon) \left( \|Mf\|_{\mathbb{L}_{2,\theta}(T)}^2 + \|g\|_{\mathbb{L}_{2,\theta}(T)}^2 \right) + \|g\|_{\mathbb{L}_{2,\theta}(T)}^2 + \|u(0)\|_{U_{2,\theta}^1}^2. \end{aligned}$$

Now it is enough to take  $\kappa = 2K/(d+1-\theta)$  and observe that (3.12) is equivalent to the condition

$$2\delta - |c| \left( \kappa + \frac{4K}{\kappa(d+1-\theta)^2} \right) = 2\delta - \frac{4|c|K}{d+1-\theta} > 0.$$

Choosing a small  $\varepsilon = \varepsilon(d, d_1, \delta, \theta, K, L) > 0$ , the lemma is proved.  $\square$

**Lemma 3.8.** *Assume that  $a_{kr}^{ij}, \sigma_{kr,m}^i$  are independent of  $x$ , and one of (3.9)-(3.11) holds. Then the assertion of Lemma 3.7 holds.*

*Proof.* We modify the proof of Lemma 3.7.

1. Denote  $S^{1j} = (s_{kr}^{1j}) = \frac{1}{2}(A^{1j} + (A^{1j})^*)$  as the symmetric part of  $A^{1j}$ . Then  $A^{1j} = S^{1j} + \frac{1}{2}H^{1j}$  and for any  $\xi \in \mathbb{R}^{d_1}$

$$\xi^* A^{1j} \xi = \xi^* S^{1j} \xi.$$

Let  $c := \theta - d$ . Note that, by integration by parts, we have

$$\int_{\mathbb{R}_+^d} u^* S^{11} u_{x^1}(x^1)^{c-1} dx = -\frac{c-1}{2} \int_{\mathbb{R}_+^d} u^* S^{11} u(x^1)^{c-2} dx = -\frac{c-1}{2} \int_{\mathbb{R}_+^d} u^* A^{11} u(x^1)^{c-2} dx$$

and hence

$$\begin{aligned} -2c \int_{\mathbb{R}_+^d} u^* A^{11} u_{x^1}(x^1)^{c-1} dx &= -2c \int_{\mathbb{R}_+^d} u^* S^{11} u_{x^1}(x^1)^{c-1} dx - c \int_{\mathbb{R}_+^d} u^* H^{11} u_{x^1}(x^1)^{c-1} dx \\ &= c(c-1) \int_{\mathbb{R}_+^d} u^* A^{11} u(x^1)^{c-2} dx - c \int_{\mathbb{R}_+^d} u^* H^{11} u_{x^1}(x^1)^{c-1} dx. \end{aligned}$$

Moreover, another usage of integration by parts gives us

$$\int_{\mathbb{R}_+^d} u^* S^{1j} u_{x^j}(x^1)^{c-1} dx = - \int_{\mathbb{R}_+^d} u_{x^j}^* S^{1j} u(x^1)^{c-1} dx = - \int_{\mathbb{R}_+^d} u^* (S^{1j})^* u_{x^j}(x^1)^{c-1} dx$$

for  $j \neq 1$ , meaning that  $\int_{\mathbb{R}_+^d} u^* S^{1j} u_{x^j}(x^1)^{c-1} dx = 0$  and

$$-2c \int_{\mathbb{R}_+^d} u^* A^{1j} u_{x^j}(x^1)^{c-1} dx = -c \int_{\mathbb{R}_+^d} u^* H^{1j} u_{x^j}(x^1)^{c-1} dx.$$

Thus the second term in (3.15) is

$$\begin{aligned} &-2c \mathbb{E} \int_0^T \int_{\mathbb{R}_+^d} (a_{kr}^{1j} u_{x^j}^r) u^k(x^1)^{c-1} dx \\ &= c(c-1) \mathbb{E} \int_0^T \int_{\mathbb{R}_+^d} u^* A^{11} u(x^1)^{c-2} dx - c \mathbb{E} \int_0^T \int_{\mathbb{R}_+^d} u^* H^{1j} u_{x^j}(x^1)^{c-1} dx, \end{aligned}$$

where the summation on  $j$  includes  $j = 1$ .

Now, as in the proof of Lemma 3.7, we have

$$\begin{aligned}
& 2\delta \|u_x\|_{\mathbb{L}_{2,\theta}(T)}^2 \\
& \leq 2\mathbb{E} \int_0^T \int_{\mathbb{R}_+^d} u_{x^i}^* (A^{ij} - \mathcal{A}^{ij}) u_{x^j} (x^1)^c dx ds \\
& \leq \mathbb{E} \int_{\mathbb{R}_+^d} |u^k(0, x)|^2 x^c dx \\
& + c(c-1)\mathbb{E} \int_0^T \int_{\mathbb{R}_+^d} a_{kr}^{11}(M^{-1}u^k)(M^{-1}u^r)(x^1)^c dx ds - c\mathbb{E} \int_0^T \int_{\mathbb{R}_+^d} (h_{kr}^{1j} u_{x^j}^r)(M^{-1}u^k)(x^1)^c dx ds \\
& + 2\mathbb{E} \int_0^T \int_{\mathbb{R}_+^d} (M^{-1}u^k)(M^f k)(x^1)^c dx ds + 2\mathbb{E} \int_0^T \int_{\mathbb{R}_+^d} (\sigma_{kr}^i u_{x^i}^r, g^k)_{\ell_2}(x^1)^c dx ds \\
& + \mathbb{E} \int_0^T \int_{\mathbb{R}_+^d} |g^k|_{\ell_2}^2 (x^1)^c dx ds. \tag{3.18}
\end{aligned}$$

Note that the terms, except the second term and the third term, in the right hand side of (3.18) are bounded by

$$\varepsilon \left( \|M^{-1}u\|_{\mathbb{L}_{2,\theta}(T)}^2 + \|u_x\|_{\mathbb{L}_{2,\theta}(T)}^2 \right) + c(\varepsilon) \left( \|Mf\|_{\mathbb{L}_{2,\theta}(T)}^2 + \|g\|_{\mathbb{L}_{2,\theta}(T)}^2 \right) + \|u(0)\|_{U_{2,\theta}^1}^2.$$

The second and the third terms will be estimated below in three steps.

2. If  $c(c-1) \geq 0$ , hence  $\theta \in (d-1, d]$ , then we have

$$\begin{aligned}
& c(c-1)\mathbb{E} \int_0^T \int_{\mathbb{R}_+^d} a_{kr}^{11}(M^{-1}u^k)(M^{-1}u^r)(x^1)^c dx ds \\
& \leq c(c-1)K^1 \|M^{-1}u\|_{\mathbb{L}_{2,\theta}(T)}^2 \leq \frac{4}{(d+1-\theta)^2} c(c-1)K^1 \|u_x\|_{\mathbb{L}_{2,\theta}(T)}^2
\end{aligned}$$

and also

$$\begin{aligned}
\left| -c \int_0^T \int_{\mathbb{R}_+^d} (h_{kr}^{1j} u_{x^j}^r)(M^{-1}u^k)(x^1)^c dx ds \right| & \leq \frac{1}{2}|c| \left( \kappa \|u_x\|_{\mathbb{L}_{2,\theta}(T)}^2 + \kappa^{-1}\beta^2 \|M^{-1}u\|_{\mathbb{L}_{2,\theta}(T)}^2 \right) \\
& \leq \frac{1}{2}|c| \left( \kappa + \frac{4\beta^2}{\kappa(d+1-\theta)^2} \right) \|u_x\|_{\mathbb{L}_{2,\theta}(T)}^2
\end{aligned}$$

for any  $\kappa > 0$ . To minimize this we take  $\kappa = 2\beta/(d+1-\theta)$ . Then

$$\left| -c \int_0^T \int_{\mathbb{R}_+^d} (h_{kr}^{1j} u_{x^j}^r)(M^{-1}u^k)(x^1)^c dx ds \right| \leq \frac{2\beta(d-\theta)}{(d+1-\theta)} \|u_x\|_{\mathbb{L}_{2,\theta}(T)}^2. \tag{3.19}$$

Thus from (3.18) we deduce

$$\begin{aligned}
& \left( 2\delta - \frac{2\beta(d-\theta)}{(d+1-\theta)} - \frac{4}{(d+1-\theta)^2} c(c-1)K^1 \right) \|u_x\|_{\mathbb{L}_{2,\theta}(T)}^2 \\
& \leq \varepsilon \|u_x\|_{\mathbb{L}_{2,\theta}(T)}^2 + c(\varepsilon) \left( \|Mf\|_{\mathbb{L}_{2,\theta}(T)}^2 + \|g\|_{\mathbb{L}_{2,\theta}(T)}^2 \right) + \|u(0)\|_{U_{2,\theta}^1}^2.
\end{aligned}$$

This and (3.17) yield the inequality (3.13) since (3.10) is equivalent to

$$2\delta - \frac{2\beta(d-\theta)}{(d+1-\theta)} - \frac{4}{(d+1-\theta)^2} c(c-1)K^1 > 0.$$

3. Again assume  $c(c-1) \geq 0$ . By (3.18) and (3.19), we have

$$\begin{aligned}
& 2\mathbb{E} \int_0^T \int_{\mathbb{R}_+^d} u_{x^i}^* (A^{ij} - \mathcal{A}^{ij}) u_{x^j} (x^1)^c dx ds \\
& \leq \mathbb{E} \int_{\mathbb{R}_+^d} |u^k(0, x)|^2 x^c dx \\
& + c(c-1) \mathbb{E} \int_0^T \int_{\mathbb{R}_+^d} (s_{d,kr}^{11} + s_{o,kr}^{11}) (M^{-1}u^k)(M^{-1}u^r)(x^1)^c dx ds \\
& + \frac{2\beta(d-\theta)}{(d+1-\theta)} \|u_x\|_{\mathbb{L}_{2,\theta}(T)}^2 + \varepsilon \|M^{-1}u\|_{\mathbb{L}_{2,\theta}(T)}^2 + c\|Mf\|_{\mathbb{L}_{2,\theta}(T)}^2.
\end{aligned}$$

By Corollary 6.2 of [13], for each  $t$ , we get

$$\begin{aligned}
& c(c-1) \int_{\mathbb{R}_+^d} s_{d,kr}^{11} (M^{-1}u^k)(M^{-1}u^r)(x^1)^c dx \\
& = c(c-1) \int_{\mathbb{R}_+^d} a_{kk}^{11} |M^{-1}u^k|^2 (x^1)^c dx \\
& \leq \frac{4(d-\theta)}{(d+1-\theta)} \int_{\mathbb{R}_+^d} a_{kk}^{ij} u_{x^i}^k u_{x^j}^k (x^1)^c dx = \frac{4(d-\theta)}{(d+1-\theta)} \int_{\mathbb{R}_+^d} u_{x^i}^* S_d^{ij} u_{x^j} (x^1)^c dx
\end{aligned}$$

and by (3.7) and (3.17),

$$\begin{aligned}
& c(c-1) \left| \int_{\mathbb{R}_+^d} s_{0,kr}^{11} M^{-1}u^k M^{-1}u^r (x^1)^c dx \right| \\
& \leq \alpha c(c-1) \int_{\mathbb{R}_+^d} |M^{-1}u|^2 (x^1)^c dx \leq \frac{4\alpha(d-\theta)}{(d+1-\theta)} \int_{\mathbb{R}_+^d} |u_x|^2 (x^1)^c dx.
\end{aligned}$$

It follows that

$$\begin{aligned}
& \mathbb{E} \int_0^T \int_{\mathbb{R}_+^d} u_{x^i}^* \left( A^{ij} - \mathcal{A}^{ij} - \frac{2(d-\theta)}{(d+1-\theta)} S_d^{ij} \right) u_{x^j} (x^1)^c dx ds \\
& \leq \frac{(d-\theta)}{(d+1-\theta)} (\beta + 2\alpha) \|u_x\|_{\mathbb{L}_{2,\theta}(T)}^2 + \varepsilon \|u_x\|_{\mathbb{L}_{2,\theta}(T)}^2 + c(\varepsilon) \left( \|Mf\|_{\mathbb{L}_{2,\theta}(T)}^2 + \|g\|_{\mathbb{L}_{2,\theta}(T)}^2 \right) + \|u(0)\|_{U_{2,\theta}^1}^2.
\end{aligned}$$

This, (3.11) and (3.17) lead to (3.13).

4. If  $c(c-1) \leq 0$ , hence  $\theta \in [d, d+1)$ , then

$$c(c-1) \mathbb{E} \int_0^T \int_{\mathbb{R}_+^d} a_{kr}^{11} (M^{-1}u^k)(M^{-1}u^r)(x^1)^c dx ds \leq \delta c(c-1) \|M^{-1}u\|_{\mathbb{L}_{2,\theta}(T)}^2;$$

for this we consider a  $d_1 \times d$  matrix  $\xi$  consisting of  $M^{-1}u$  as the first column and zeros for the rest and apply the condition (2.3). Next, as before, we have

$$\left| -c \mathbb{E} \int_0^T \int_{\mathbb{R}_+^d} (h_{kr}^{1j} u_{x^j}^r) (M^{-1}u^k) (x^1)^c dx ds \right| \leq \frac{1}{2} c \left( \kappa \|u_x\|_{\mathbb{L}_{2,\theta}(T)}^2 + \kappa^{-1} \beta^2 \|M^{-1}u\|_{\mathbb{L}_{2,\theta}(T)}^2 \right)$$

and hence

$$\begin{aligned}
& 2\delta \|u_x\|_{\mathbb{L}_{2,\theta}(T)}^2 - \frac{1}{2}c \left( \kappa \|u_x\|_{\mathbb{L}_{2,\theta}(T)}^2 + \kappa^{-1}\beta^2 \|M^{-1}u\|_{\mathbb{L}_{2,\theta}(T)}^2 \right) - \delta c(c-1) \|M^{-1}u\|_{\mathbb{L}_{2,\theta}(T)}^2 \\
& \leq \varepsilon \left( \frac{4}{(d+1-\theta)^2} + d^2L \right) \|u_x\|_{\mathbb{L}_{2,\theta}(T)}^2 + c(\varepsilon) \left( \|Mf\|_{\mathbb{L}_{2,\theta}(T)}^2 + K \|g\|_{\mathbb{L}_{2,\theta}(T)}^2 \right) \\
& \quad + \|g\|_{\mathbb{L}_{2,\theta}(T)}^2 + \|u(0)\|_{\dot{U}_{2,\theta}^1}^2.
\end{aligned} \tag{3.20}$$

As we take

$$\kappa = \frac{\beta^2}{2\delta(1-c)},$$

the terms with  $\|M^{-1}u\|_{\mathbb{L}_{2,\theta}(T)}^2$  in the left hand side of (3.20) are canceled out. Now, (3.9) which is equivalent to  $2\delta - \frac{c\beta^2}{4\delta(1-c)} > 0$  gives us (3.13). The lemma is proved.  $\square$

The following lemma with Definition 3.3 will lead to an a priori estimate:

**Lemma 3.9.** *Let  $\mu \in \mathbb{R}$ ,  $f \in M^{-1}\mathbb{H}_{2,\theta}^\mu(T)$ ,  $g \in \mathbb{H}_{2,\theta}^{\mu+1}(T, \ell_2)$ ,  $u(0) \in U_{2,\theta}^{\mu+2}$  and  $u \in M\mathbb{H}_{2,\theta}^{\mu+1}(T)$  be a solution of the problem (3.5) on  $[0, T] \times \mathbb{R}_+^d$ , then  $u \in M\mathbb{H}_{2,\theta}^{\mu+2}(T)$  and*

$$\|M^{-1}u\|_{\mathbb{H}_{2,\theta}^{\mu+2}(T)} \leq c \left( \|M^{-1}u\|_{\mathbb{H}_{2,\theta}^{\mu+1}(T)} + \|Mf\|_{\mathbb{H}_{2,\theta}^\mu(T)} + \|g\|_{\mathbb{H}_{2,\theta}^{\mu+1}(T, \ell_2)} + \|u(0)\|_{U_{2,\theta}^{\mu+2}} \right), \tag{3.21}$$

where  $c = c(d, d_1, \mu, \theta, \delta, K, L)$ .

*Proof.* By Lemma 3.1 (ii) and (2.1), we have

$$\begin{aligned}
\|M^{-1}u\|_{\mathbb{H}_{2,\theta}^{\mu+2}(T)}^2 & \leq c \sum_n e^{n(\theta-2)} \|u(t, e^n x) \zeta(x)\|_{\mathbb{H}_2^{\mu+2}(T)}^2 \\
& = c \sum_n e^{n\theta} \|u(e^{2n}t, e^n x) \zeta(x)\|_{\mathbb{H}_2^{\mu+2}(e^{-2n}T)}^2 \\
& \leq c \sum_n e^{n\theta} \|(u(e^{2n}t, e^n x) \zeta(x))_{xx}\|_{\mathbb{H}_2^\mu(e^{-2n}T)}^2.
\end{aligned}$$

Denote

$$v_n(\omega, t, x) = u(\omega, e^{2n}t, e^n x) \zeta(x), \quad (a_n)^{ij}_{kr}(\omega, t) = a_{kr}^{ij}(\omega, e^{2n}t), \quad (\sigma_n)^i_{kr}(\omega, t) = \sigma_{kr}^i(\omega, e^{2n}t)$$

$$A_n^{ij} = ((a_n)^{ij}_{kr}), \quad \Sigma_n^i = ((\sigma_n)^i_{kr}).$$

Then, since  $v_n$  has compact support in  $\mathbb{R}_+^d$ , we can regard it as a distribution defined on the whole space. Thus  $v_n$  is in  $\mathbb{H}_2^{\mu+1}(e^{-2n}T)$  and satisfies

$$dv_n = (A_n^{ij}(v_n)_{x^i x^j} + f_n) dt + ((\Sigma_n^i)_m(v_n)_{x^i} + (g_n)_m) d(e^{-n}w_{e^{2n}t}^m), \quad v_n(0, x) = \zeta(x)u_0(e^n x),$$

where  $(\Sigma_n^i)_m = ((\sigma_n)^i_{kr,m})$  and

$$f_n = -2e^n A_n^{ij} u_{x^i}(e^{2n}t, e^n x) \zeta_{x^j}(x) - A_n^{ij} u(e^{2n}t, e^n x) \zeta_{x^i x^j}(x) + e^{2n} f(e^{2n}t, e^n x) \zeta(x),$$

$$(g_n)_m = -(\Sigma_n^i)_m u(e^{2n}t, e^n x) \zeta_{x^i}(x) + e^n g_m(e^{2n}t, e^n x) \zeta(x).$$

Then, by Theorem 2.3,  $v_n$  is in  $\mathcal{H}_2^{\mu+2}(e^{-2n}T)$  and

$$\|(v_n)_{xx}\|_{\mathbb{H}_2^\mu(e^{-2n}T)}^2 \leq c(d, d_1, \mu, \delta, K, L) \left( \|f_n\|_{\mathbb{H}_2^\mu(e^{-2n}T)}^2 + \|g_n\|_{\mathbb{H}_2^{\mu+1}(e^{-2n}T, \ell_2)}^2 + \|\zeta(x)u_0(e^n x)\|_{U_2^{\mu+2}}^2 \right).$$

Thus, by (3.3) and Lemma 3.1,

$$\begin{aligned} & \sum_n e^{n\theta} \|(u(e^{2n}t, e^n x)\zeta(x))_{xx}\|_{\mathbb{H}_2^\mu(e^{-2n}T)}^2 \\ & \leq c \sum_n \left[ e^{n\theta} \|u_x(t, e^n \cdot)\zeta_x\|_{\mathbb{H}_2^\mu(T)}^2 \right] + c \sum_n e^{n(\theta-2)} \|u(t, e^n \cdot)\zeta_{xx}\|_{\mathbb{H}_2^\mu(T)}^2 \\ & \quad + c \sum_n e^{n(\theta+2)} \|f(t, e^n \cdot)\zeta\|_{\mathbb{H}_2^\mu(T)}^2 + c \sum_n \left[ e^{n(\theta-2)} \|u(t, e^n \cdot)\zeta_x\|_{\mathbb{H}_2^\mu(T)}^2 \right] \\ & \quad + c \sum_n e^{n\theta} \|g(t, e^n x)\zeta\|_{\mathbb{H}_2^\mu(T, \ell_2)}^2 + \sum_n e^{n\theta} \|u_0(t, e^n x)\zeta\|_{U_2^{\mu+2}}^2 \\ & \leq c \|M^{-1}u\|_{\mathbb{H}_{2,\theta}^{\mu+1}(T)}^2 + c \|Mf\|_{\mathbb{H}_{2,\theta}^\mu(T)}^2 + c \|g\|_{\mathbb{H}_{2,\theta}^{\mu+1}(T, \ell_2)}^2 + c \|u_0\|_{U_{2,\theta}^{\mu+2}}^2. \end{aligned}$$

The lemma is proved.  $\square$

From this point on we assume the following:

**Assumption 3.10.** There exists a constant  $N > 0$ , **independent of  $x$** , so that for each fixed  $x$ ,  $(A^{ij}(\cdot, \cdot, x), \Sigma^i(\cdot, \cdot, x), \theta)$  is admissible with constant  $N$ .

First, we prove our results for the problem (3.13) with the coefficients independent of  $x$ .

**Theorem 3.11.** *Suppose Assumptions 2.2 and 3.10 hold. Also assume that  $A^{ij}, \Sigma^i$  are independent of  $x$ . Then for any  $f \in M^{-1}\mathbb{H}_{2,\theta}^\gamma(T)$ ,  $g \in \mathbb{H}_{2,\theta}^{\gamma+1}(T, \ell_2)$ ,  $u_0 \in U_{2,\theta}^{\gamma+2}$ , the problem (3.5) admits a unique solution  $u \in \mathfrak{H}_{2,\theta}^{\gamma+2}(T)$ , and for this solution*

$$\|u\|_{\mathfrak{H}_{2,\theta}^{\gamma+2}(T)} \leq c \left( \|Mf\|_{\mathbb{H}_{2,\theta}^\gamma(T)} + \|g\|_{\mathbb{H}_{2,\theta}^{\gamma+1}(T, \ell_2)} + \|u_0\|_{U_{2,\theta}^{\gamma+2}} \right), \quad (3.22)$$

where  $c = c(d, d_1, \delta, \theta, K^j, L)$ .

*Proof.* 1. By Theorem 3.3 in [16], for each  $k$ , the single equation

$$du^k = (\delta \Delta u^k + f^k)dt + g^k dw_t, \quad u^k(0) = u_0^k$$

has a solution  $u^k \in \mathfrak{H}_{2,\theta}^{\gamma+2}(T)$ . As in the proof of Theorem 2.3 we only need to show that the estimate (3.22) holds given that a solution already exists. Also by Lemma 3.1 and (3.4) it is enough to show

$$\|M^{-1}u\|_{\mathbb{H}_{2,\theta}^{\gamma+2}(T)} \leq c \left( \|Mf\|_{\mathbb{H}_{2,\theta}^\gamma(T)} + \|g\|_{\mathbb{H}_{2,\theta}^{\gamma+1}(T, \ell_2)} + \|u_0\|_{U_{2,\theta}^{\gamma+2}} \right). \quad (3.23)$$

2. Assume  $\gamma \geq 0$ . By Theorem 2.9 in [16], for any nonnegative integer  $n \geq \gamma$ , the set

$$\mathfrak{H}_{2,\theta}^n(T) \cap \bigcup_{N=1}^{\infty} L_2(\Omega, C([0, T], C_0^n((1/N, N) \times \{x' : |x'| < N\})))$$



is dense in  $\mathfrak{H}_{2,\theta}^\gamma(T)$  and we may assume that  $u$  is sufficiently smooth in  $x$  and vanishes near the boundary. Let  $m$  be an integer so that  $\gamma + 1 - m \leq 0$ . Then by applying Lemma 3.9 with  $\mu = \gamma, \gamma - 1, \dots, \gamma - m$  in order,

$$\|M^{-1}u\|_{\mathbb{H}_{2,\theta}^{\gamma+2}(T)} \leq c \left( \|M^{-1}u\|_{\mathbb{H}_{2,\theta}^{\gamma+1-m}(T)} + \|Mf\|_{\mathbb{H}_{2,\theta}^\gamma(T)} + \|g\|_{\mathbb{H}_{2,\theta}^{\gamma+1}(T,\ell_2)} + \|u_0\|_{U_{2,\theta}^{\gamma+2}} \right).$$

Thus to get (3.23) it is enough to use the fact  $\|\cdot\|_{H_{2,\theta}^{\gamma+1-m}} \leq \|\cdot\|_{L_{2,\theta}}$  and the inequality (3.13).

3. Assume  $\gamma \in [-1, 0)$ , i.e.,  $\gamma + 1 \geq 0$ . Recall  $\mathcal{L}u := (M^2\Delta + 2MD_1)u = (x^1)^2\Delta u + 2x^1u_{x^1}$ . We have  $\bar{f} := \mathcal{L}^{-1}f \in M^{-1}\mathbb{H}_{2,\theta}^{\gamma+2}(T)$ ,  $\bar{g} := \mathcal{L}^{-1}g \in \mathbb{H}_{2,\theta}^{\gamma+3}(T, \ell_2)$  and  $\bar{u}_0 := \mathcal{L}^{-1}u_0 \in U_{2,\theta}^{\gamma+4}$ . If  $\bar{u} \in \mathfrak{H}_{2,\theta}^{\gamma+4}(T)$  is the solution of the problem

$$d\bar{u} = (A^{ij}\bar{u}_{x^i x^j} + \bar{f})dt + (\Sigma_m^i \bar{u}_{x^i} + \bar{g}_m)dw_t^m, \quad \bar{u}(0) = \bar{u}_0$$

with  $\Sigma_m^i = (\sigma_{kr,m}^i)$ , then for  $v = \mathcal{L}\bar{u}$  we have  $v \in \mathfrak{H}_{2,\theta}^{\gamma+2}(T)$  and

$$\begin{aligned} dv &= (A^{ij}v_{x^i x^j} + f - 2(A^{1i} + A^{i1})(\bar{u}_{x^1 x^i} + x^1 \Delta \bar{u}_{x^i}) - 2A^{11} \Delta \bar{u}) dt \\ &\quad + (\Sigma_m^i v_{x^i} + g_m - 2\Sigma_m^1(\bar{u}_{x^1} + x^1 \Delta \bar{u})) dw_t^m, \quad t > 0 \\ v(0) &= u_0. \end{aligned}$$

Since  $\bar{u}_{x^1 x^i} + x^1 \Delta \bar{u}_{x^i} = M^{-1}\mathcal{L}(\bar{u}_{x^i}) \in M^{-1}\mathbb{H}_{2,\theta}^{\gamma+1}(T)$ ,  $\bar{u}_{x^i x^j} \in M^{-1}\mathbb{H}_{2,\theta}^{\gamma+2}(T)$ ,  $\bar{u}_{x^1} \in \mathbb{H}_{2,\theta}^{\gamma+3}(T)$ , and  $\gamma + 1 \geq 0$ , we can find a  $\tilde{u} \in \mathfrak{H}_{2,\theta}^{\gamma+3}(T)$  as the solution of

$$\begin{aligned} d\tilde{u} &= (A^{ij}\tilde{u}_{x^i x^j} - 2(A^{1i} + A^{i1})(\bar{u}_{x^1 x^i} + x^1 \Delta \bar{u}_{x^i}) - 2A^{11} \Delta \bar{u}) dt \\ &\quad + (\Sigma_m^i \tilde{u}_{x^i} - 2\Sigma_m^1(\bar{u}_{x^1} + x^1 \Delta \bar{u})) dw_t^m, \\ \tilde{u}(0) &= 0. \end{aligned}$$

Then  $u := v - \tilde{u} \in \mathfrak{H}_{2,\theta}^{\gamma+2}(T)$  satisfies (3.5) and estimate (3.23) follows from the formula defining  $v, \tilde{u}$  and the fact that

$$\|M^{-1}v\|_{\mathbb{H}_{2,\theta}^{\gamma+2}(T)} \leq c\|M^{-1}\bar{u}\|_{\mathbb{H}_{2,\theta}^{\gamma+4}(T)}, \quad \|M^{-1}\tilde{u}\|_{\mathbb{H}_{2,\theta}^{\gamma+2}(T)} \leq c\|M^{-1}\bar{u}\|_{\mathbb{H}_{2,\theta}^{\gamma+4}(T)}.$$

Now, we pass to proving the uniqueness of the solution in the space  $\mathfrak{H}_{2,\theta}^{\gamma+2}(T)$ . Let  $u \in \mathfrak{H}_{2,\theta}^{\gamma+2}(T)$  be a solution with  $f = 0, g = 0, u_0 = 0$ . We claim that  $u \equiv 0$ . For this we just show that  $u \in \mathfrak{H}_{2,\theta}^{\gamma+3}(T)$ , or equivalently  $v = \mathcal{L}^{-1}u \in \mathfrak{H}_{2,\theta}^{\gamma+5}(T)$  since we have already proved the uniqueness in  $\mathfrak{H}_{2,\theta}^{\gamma+3}(T)$  at step 2; recall  $\gamma + 1 \geq 0$ . In fact, since  $u \in \mathfrak{H}_{2,\theta}^{\gamma+2}(T)$  at least, we have  $v \in \mathfrak{H}_{2,\theta}^{\gamma+4}(T)$  and

$$dv = (A^{ij}v_{x^i x^j} + \bar{f})dt + (\Sigma_m^i v_{x^i} + \bar{g}_m)dw_t^m,$$

where

$$\bar{f} = A^{ij}\mathcal{L}^{-1}(u_{x^i x^j}) - A^{ij}(\mathcal{L}^{-1}u)_{x^i x^j}, \quad \bar{g}_m = \Sigma_m^i \mathcal{L}^{-1}(u_{x^i}) - \Sigma_m^i(\mathcal{L}^{-1}u)_{x^i}.$$

However, we observe

$$\begin{aligned} \mathcal{L}\bar{f} &= A^{ij}(u_{x^i x^j} - \mathcal{L}((\mathcal{L}^{-1}u)_{x^i x^j})) \\ &= 2(A^{1i} + A^{i1})M^{-1}u_{x^i} - 8A^{11}M^{-2}u + 2A^{11}\Delta(\mathcal{L}^{-1}u) \in M^{-1}\mathbb{H}_{2,\theta}^{\gamma+1}(T), \\ \mathcal{L}\bar{g} &= \Sigma^i(u_{x^i} - \mathcal{L}((\mathcal{L}^{-1}u)_{x^i})) = 2\Sigma^1 M^{-1}u \in \mathbb{H}_{2,\theta}^{\gamma+2}(T, \ell_2). \end{aligned} \tag{3.24}$$

Thus  $\bar{f} \in M^{-1}\mathbb{H}_{2,\theta}^{\gamma+3}(T)$  and  $\bar{g} \in \mathbb{H}_{2,\theta}^{\gamma+4}(T, \ell_2)$ . Consequently,  $v \in \mathfrak{H}_{2,\theta}^{\gamma+5}(T)$  and  $u \equiv 0$ .

4. The case  $\gamma \in [-n-1, -n)$  with  $n \in \{1, 2, \dots\}$  is treated similarly. The theorem is proved.  $\square$

Now, we prove our results for the problem (1.1) with variable coefficients. For  $n \in \mathbb{Z}$ ,  $\mu \in (0, 1]$  and  $k = 0, 1, 2, \dots$ , we define

$$[u]_k^{(n)} = \sup_{\substack{x \in \mathbb{R}_+^d \\ |\beta|=k}} (x^1)^{k+n} |D^\beta u(x)|, \quad (3.25)$$

$$[u]_{k+\mu}^{(n)} = \sup_{\substack{x, y \in \mathbb{R}_+^d \\ |\beta|=k}} (x^1 \wedge y^1)^{k+\mu+n} \frac{|D^\beta u(x) - D^\beta u(y)|}{|x - y|^\mu}, \quad (3.26)$$

$$|u|_k^{(n)} = \sum_{j=0}^k [u]_j^{(n)}, \quad |u|_{k+\mu}^{(n)} = |u|_k^{(n)} + [u]_{k+\mu}^{(n)}.$$

Here is the main result of this section.

**Theorem 3.12.** *Let Assumptions 2.2 and 3.10 hold, and*

$$|a_{kr}^{ij}(t, \cdot)|_{|\gamma|_+}^{(0)} + |b_{kr}^i(t, \cdot)|_{|\gamma|_+}^{(1)} + |c_{kr}(t, \cdot)|_{|\gamma|_+}^{(2)} + |\sigma_{kr}^i(t, \cdot)|_{|\gamma+1|_+}^{(0)} + |\nu_{kr}(t, \cdot)|_{|\gamma+1|_+}^{(1)} \leq L \quad (3.27)$$

and

$$|a_{kr}^{ij}(t, x) - a_{kr}^{ij}(t, y)| + |\sigma_{kr}^i(t, x) - \sigma_{kr}^i(t, y)|_{\ell_2} + |Mb_{kr}^i(t, x)| + |M^2 c_{kr}(t, x)| + |M \nu_{kr}(t, x)|_{\ell_2} < \kappa \quad (3.28)$$

for all  $x, y \in \mathbb{R}_+^d$  with  $|x - y| \leq x^1 \wedge y^1$ . Then there exists  $\kappa_0 = \kappa_0(d, d_1, \theta, \delta, \gamma, K, L)$  so that if  $\kappa \leq \kappa_0$ , then for any  $f \in M^{-1}\mathbb{H}_{2,\theta}^\gamma(T)$ ,  $g \in \mathbb{H}_{2,\theta}^{\gamma+1}(T, \ell_2)$  and  $u_0 \in U_{2,\theta}^{\gamma+2}$ , the problem (1.1) defined on  $\Omega \times [0, T] \times \mathbb{R}_+^d$  admits a unique solution  $u \in \mathfrak{H}_{2,\theta}^{\gamma+2}(T)$ , and it holds that

$$\|M^{-1}u\|_{\mathbb{H}_{2,\theta}^{\gamma+2}(T)} \leq c \left( \|Mf\|_{\mathbb{H}_{2,\theta}^\gamma(T)} + \|g\|_{\mathbb{H}_{2,\theta}^{\gamma+1}(T, \ell_2)} + \|u_0\|_{U_{2,\theta}^{\gamma+2}} \right) \quad (3.29)$$

where  $c = c(d, d_1, \delta, K^j, L)$ .

*Remark 3.13.* See Remark 4.7(i) for the better understanding of the condition (3.28).

*Remark 3.14.* Since  $C_0^\infty$  is dense in  $H_{p,\theta}^\gamma$ , zero boundary condition is implicitly imposed in Theorem 3.12 (and in Theorem 4.8 below).

To prove Theorem 3.12 we use the following three lemmas taken from [8].

**Lemma 3.15.** *Let constants  $C, \delta$  be in  $(0, \infty)$ , and  $q$  be the smallest integer such that  $|\gamma| + 2 \leq q$ .*

(i) *Let  $\eta_n \in C^\infty(\mathbb{R}_+^d)$ ,  $n = 1, 2, \dots$ , satisfy*

$$\sum_n M^{|\alpha|} |D^\alpha \eta_n| \leq C \quad \text{in } \mathbb{R}_+^d \quad (3.30)$$

for any multi-index  $\alpha$  such that  $0 \leq |\alpha| \leq q$ . Then for any  $u \in H_{p,\theta}^\gamma$

$$\sum_n \|\eta_n u\|_{H_{p,\theta}^\gamma}^p \leq NC^p \|u\|_{H_{p,\theta}^\gamma}^p,$$

where the constant  $N$  is independent of  $u$ ,  $\theta$ , and  $C$ .

(ii) If, in addition to the condition in (i),  $\sum_n \eta_n^2 \geq \delta$  on  $\mathbb{R}_+^d$ , then for any  $u \in H_{p,\theta}^\gamma$ ,

$$\|u\|_{H_{p,\theta}^\gamma}^p \leq N \sum_n \|\eta_n u\|_{H_{p,\theta}^\gamma}^p, \quad (3.31)$$

where the constant  $N$  is independent of  $u$  and  $\theta$ .

The reason that the first inequality in (3.32) below is written for  $\eta_n^4$  (not for  $\eta_n^2$  as in the above lemma) is to have the possibility to apply Lemma 3.15 to  $\eta_n^2$ . Also, note  $\sum a^2 \leq (\sum |a|)^2$ .

**Lemma 3.16.** For each  $\varepsilon > 0$  and  $q = 1, 2, \dots$ , there exist non-negative functions  $\eta_n \in C_0^\infty(\mathbb{R}_+^d)$ ,  $n = 1, 2, \dots$  such that (i) on  $\mathbb{R}_+^d$  for each multi-index  $\alpha$  with  $1 \leq |\alpha| \leq q$  we have

$$\sum_n \eta_n^4 \geq 1, \quad \sum_n \eta_n \leq N(d), \quad \sum_n M^{|\alpha|} |D^\alpha \eta_n| \leq \varepsilon; \quad (3.32)$$

(ii) for any  $n$  and  $x, y \in \text{supp } \eta_n$  we have  $|x - y| \leq N(x^1 \wedge y^1)$ , where  $N = N(d, q, \varepsilon) \in [1, \infty)$ .

**Lemma 3.17.** Let  $p \in (1, \infty)$ ,  $\gamma, \theta \in \mathbb{R}$ . Then there exists a constant  $N = N(\gamma, |\gamma|_+, p, d)$  such that if  $f \in H_{p,\theta}^\gamma$  and  $a$  is a function with the finite norm  $|a|_{|\gamma|_+}^{(0)}$ , then

$$\|af\|_{H_{p,\theta}^\gamma} \leq N |a|_{|\gamma|_+}^{(0)} \|f\|_{H_{p,\theta}^\gamma}. \quad (3.33)$$

In addition,

(i) if  $\gamma = 0, 1, 2, \dots$ , then

$$\|af\|_{H_{p,\theta}^\gamma} \leq N_1 \sup_{\mathbb{R}_+^d} |a| \|f\|_{H_{p,\theta}^\gamma} + N_2 \|f\|_{H_{p,\theta}^{\gamma-1}} \sup_{\mathbb{R}_+^d} \sup_{1 \leq |\alpha| \leq \gamma} |M^{|\alpha|} D^\alpha a|, \quad (3.34)$$

where, obviously, one can take  $N_1 = 1$  and  $N_2 = 0$  if  $\gamma = 0$ .

(ii) if  $\gamma$  is not integer, then

$$\|af\|_{H_{p,\theta}^\gamma} \leq N (\sup_{\mathbb{R}_+^d} |a|)^s (|a|_{|\gamma|_+}^{(0)})^{1-s} \|f\|_{H_{p,\theta}^\gamma}, \quad (3.35)$$

where  $s := 1 - \frac{|\gamma|}{|\gamma|_+} > 0$ .

The same assertions hold true for  $\ell_2$ -valued  $a$ .

**Proof of Theorem 3.12** We proceed as in Theorem 2.16 of [7], where the theorem is proved for single equations. As usual, for simplicity, we assume  $u_0 = 0$  (see the proof of Theorem 5.1 in [12]). Also having the method of continuity in mind, we convince ourselves that to prove the theorem it suffices to show that there exists  $\kappa_0$  such that the a priori estimate (3.29) holds given that the solution already exists and  $\kappa \leq \kappa_0$ . We divide the proof into 6 cases. The reason for this is that if  $\gamma$  is not an integer we use (3.35) and if  $\gamma$  is a non-negative integer we use (3.34), but if  $\gamma$  is a negative integer we use the somewhat different approaches used in [7].

**Case 1:**  $|\gamma| \notin \{0, 1, 2, \dots\}$ . Take the least integer  $q \geq |\gamma| + 4$ . Also take an  $\varepsilon \in (0, 1)$  which will be specified later, and take a sequence of functions  $\eta_n$ ,  $n = 1, 2, \dots$  from Lemma 3.16 corresponding to  $\varepsilon, q$ . Then by Lemma 3.15, we have

$$\|M^{-1}u\|_{\mathbb{H}_{2,\theta}^{\gamma+2}(T)}^2 \leq N \sum_{n=1}^{\infty} \|M^{-1}u\eta_n^2\|_{\mathbb{H}_{2,\theta}^{\gamma+2}(T)}^2. \quad (3.36)$$

For any  $n$  let  $x_n$  be a point in  $\text{supp } \eta_n$  and  $a_{kr,n}^{ij}(t) = a_{kr}^{ij}(t, x_n)$ ,  $\sigma_{kr,n,m}^i(t) = \sigma_{kr,m}^i(t, x_n)$ . From (1.1), we easily have

$$d(u^k \eta_n^2) = (a_{kr,n}^{ij}(u^r \eta_n^2)_{x^i x^j} + M^{-1} f_n^k) dt + (\sigma_{kr,n,m}^i(u^r \eta_n^2)_{x^i} + g_{n,m}^k) dw_t^m,$$

where

$$\begin{aligned} f_n^k &= (a_{kr}^{ij} - a_{kr,n}^{ij}) \eta_n^2 M u_{x^i x^j}^r - 2a_{kr,n}^{ij} M (\eta_n^2)_{x^i} u_{x^j}^r - a_{kr,n}^{ij} M^{-1} u^r M^2 (\eta_n^2)_{x^i x^j} \\ &\quad + \eta_n^2 M b_{kr}^i u_{x^i}^r + \eta_n^2 M^2 c_{kr} M^{-1} u^r + M f_n^k \eta_n^2, \\ g_{n,m}^k &= (\sigma_{kr,m}^i - \sigma_{kr,n,m}^i) \eta_n^2 u_{x^i}^r - \sigma_{kr,n,m}^i M^{-1} u^r M (\eta_n^2)_{x^i} + M \nu_{kr,m} M^{-1} u^r \eta_n^2 + g_m^k \eta_n^2. \end{aligned}$$

By Theorem 3.11, for each  $n$ ,

$$\|M^{-1}u\eta_n^2\|_{\mathbb{H}_{2,\theta}^{\gamma+2}(T)}^2 \leq N(\|f_n\|_{\mathbb{H}_{2,\theta}^{\gamma}(T)}^2 + \|g_n\|_{\mathbb{H}_{2,\theta}^{\gamma+1}(T,\ell_2)}^2) \quad (3.37)$$

and by (3.35),

$$\|(a_{kr}^{ij} - a_{kr,n}^{ij}) \eta_n^2 M u_{x^i x^j}^r\|_{\mathbb{H}_{2,\theta}^{\gamma}(T)} \leq N \|\eta_n M u_{xx}\|_{\mathbb{H}_{2,\theta}^{\gamma}(T)} \sup_{\omega, t, x} |(a_{kr}^{ij} - a_{kr,n}^{ij}) \eta_n|^s, \quad (3.38)$$

where  $s > 0$  is a constant depending only on  $\gamma$  and  $|\gamma|_+$ .

By Lemma 3.16 (ii), for each  $n$  and  $x, y \in \text{supp } \eta_n$  we have  $|x - y| \leq N(\varepsilon)(x^1 \wedge y^1)$ , where  $N(\varepsilon) = N(d, q, \varepsilon)$ , and we can easily fix points  $x_i$  lying on the straight segment connecting  $x$  and  $y$  and including  $x$  and  $y$  so that the number of points are not more than  $N(\varepsilon) + 2 \leq 3N(\varepsilon)$  and  $|x_i - x_{i+1}| \leq x_i^1 \wedge x_{i+1}^1$ . It follows from our assumptions

$$\sup_{\omega, t, x} |(a_{kr}^{ij} - a_{kr,n}^{ij}) \eta_n| \leq 3N(\varepsilon) \kappa.$$

We substitute this to (3.38) and get

$$\|(a_{kr}^{ij} - a_{kr,n}^{ij}) \eta_n^2 M u_{x^i x^j}^r\|_{\mathbb{H}_{2,\theta}^{\gamma}(T)} \leq NN(\varepsilon) \kappa^s \|\eta_n M u_{xx}\|_{\mathbb{H}_{2,\theta}^{\gamma}(T)}.$$

Similarly,

$$\begin{aligned} &\|\eta_n^2 M b_{kr}^i u_{x^i}^r\|_{\mathbb{H}_{2,\theta}^{\gamma}(T)} + \|\eta_n^2 M^2 c_{kr} M^{-1} u^r\|_{\mathbb{H}_{2,\theta}^{\gamma}(T)} + \|(\sigma_{kr}^i - \sigma_{kr,n}^i) \eta_n^2 u_{x^i}^r\|_{\mathbb{H}_{2,\theta}^{\gamma+1}(T,\ell_2)} \\ &+ \|\eta_n^2 M \nu_{kr} M^{-1} u^r\|_{\mathbb{H}_{2,\theta}^{\gamma+1}(T,\ell_2)} \leq NN(\varepsilon) \kappa^s \left( \|\eta_n u_x\|_{\mathbb{H}_{2,\theta}^{\gamma+1}(T)} + \|\eta_n M^{-1} u\|_{\mathbb{H}_{2,\theta}^{\gamma+1}(T)} \right). \end{aligned}$$

Coming back to (3.37) and (3.36) and using Lemma 3.15, we conclude

$$\|M^{-1}u\|_{\mathbb{H}_{2,\theta}^{\gamma+2}(T)}^2 \leq NN(\varepsilon) \kappa^{2s} \left( \|M u_{xx}\|_{\mathbb{H}_{2,\theta}^{\gamma}(T)}^2 + \|u_x\|_{\mathbb{H}_{2,\theta}^{\gamma+1}(T)}^2 + \|M^{-1}u\|_{\mathbb{H}_{2,\theta}^{\gamma+1}(T)}^2 \right)$$

$$+ NC^2 \left( \|u_x\|_{\mathbb{H}_{2,\theta}^\gamma(T)}^2 + \|M^{-1}u\|_{\mathbb{H}_{2,\theta}^{\gamma+1}(T)}^2 \right) + N \left( \|Mf\|_{\mathbb{H}_{2,\theta}^\gamma(T)}^2 + \|g\|_{\mathbb{H}_{2,\theta}^{\gamma+1}(T,\ell_2)}^2 \right), \quad (3.39)$$

where

$$C = \sup_{\mathbb{R}_+^d} \sup_{|\alpha| \leq q-2} \sum_{n=1}^{\infty} M^{|\alpha|} (|D^\alpha(M(\eta_n^2)_x)| + |D^\alpha(M^2(\eta_n^2)_{xx})|).$$

By construction, we have  $C \leq N\varepsilon$ . Furthermore (see Lemma 3.1)

$$\|u_x\|_{H_{2,\theta}^{\gamma+1}} \leq N\|M^{-1}u\|_{H_{2,\theta}^{\gamma+2}}, \quad \|Mu_{xx}\|_{H_{2,\theta}^\gamma} \leq N\|M^{-1}u\|_{H_{2,\theta}^{\gamma+2}}. \quad (3.40)$$

Hence (3.39) yields

$$\|M^{-1}u\|_{\mathbb{H}_{2,\theta}^{\gamma+2}(T)}^2 \leq N_1(N(\varepsilon)\kappa^{2s} + \varepsilon^2)\|M^{-1}u\|_{\mathbb{H}_{2,\theta}^{\gamma+2}(T)}^2 + N \left( \|Mf\|_{\mathbb{H}_{2,\theta}^\gamma(T)}^2 + \|g\|_{\mathbb{H}_{2,\theta}^{\gamma+1}(T,\ell_2)}^2 \right).$$

Finally, to get a priori estimate (3.29) it's enough to choose first  $\varepsilon$  and then  $\kappa_0$  so that  $N_1(N(\varepsilon)\kappa^{2s} + \varepsilon^2) \leq 1/2$  for  $\kappa \leq \kappa_0$ .

**Case 2:**  $\gamma = 0$ . Proceed as in Case 1 with  $\varepsilon = 1$  and arrive at (3.37) which is

$$\|M^{-1}u\eta_n^2\|_{\mathbb{H}_{2,\theta}^2(T)}^2 \leq N \left( \|f_n\|_{\mathbb{L}_{2,\theta}(T)}^2 + \|g_n\|_{\mathbb{H}_{2,\theta}^1(T,\ell_2)}^2 \right).$$

Notice that (3.38) holds with  $s = 1$  (since  $\gamma = 0$ ). Also by (3.34),

$$\begin{aligned} \|(\sigma_{kr}^i - \sigma_{kr,n}^i)\eta_n^2 u_{x^i}\|_{\mathbb{H}_{2,\theta}^1(T,\ell_2)} &\leq N \sup_{\omega,t,x} |(\sigma_{kr}^i - \sigma_{kr,n}^i)\eta_n|_{\ell_2} \|\eta_n u_x\|_{\mathbb{H}_{2,\theta}^1(T)} + N\|\eta_n u_x\|_{\mathbb{L}_{2,\theta}(T)} \\ &\leq N\kappa\|\eta_n u_x\|_{\mathbb{H}_{2,\theta}^1(T)} + N\|\eta_n u_x\|_{\mathbb{L}_{2,\theta}(T)}. \end{aligned} \quad (3.41)$$

From this point, by following the arguments in Case 1, one gets

$$\|M^{-1}u\|_{\mathbb{H}_{2,\theta}^2(T)} \leq N_1\kappa\|M^{-1}u\|_{\mathbb{H}_{2,\theta}^2(T)} + N\|M^{-1}u\|_{\mathbb{H}_{2,\theta}^1(T)} + N\|Mf\|_{\mathbb{L}_{2,\theta}(T)} + N\|g\|_{\mathbb{H}_{2,\theta}^1(T)}.$$

Thus, if  $N_1\kappa_0 \leq 1/2$  and  $\kappa \leq \kappa_0$ , then we have

$$\|M^{-1}u\|_{\mathbb{H}_{2,\theta}^2(T)} \leq N\|M^{-1}u\|_{\mathbb{H}_{2,\theta}^1(T)} + N\|Mf\|_{\mathbb{L}_{2,\theta}(T)} + N\|g\|_{\mathbb{H}_{2,\theta}^1(T,\ell_2)}. \quad (3.42)$$

Next, if necessary, by reducing  $\kappa_0$  (note that we are free to do this) we will estimate the norm  $\|M^{-1}u\|_{\mathbb{H}_{2,\theta}^1(T)}$ . Take an  $\varepsilon \in (0, 1)$  which will be specified later and proceed as in Case 1 and write (3.36) and (3.37) for  $\gamma = -1$ . The latter is

$$\|M^{-1}u\eta_n^2\|_{\mathbb{H}_{2,\theta}^1(T)}^2 \leq N \left( \|f_n\|_{\mathbb{H}_{2,\theta}^{-1}(T)}^2 + \|g_n\|_{\mathbb{L}_{2,\theta}(T,\ell_2)}^2 \right).$$

Using the fact  $\|f_n\|_{\mathbb{H}_{2,\theta}^{-1}(T)} \leq \|f_n\|_{\mathbb{L}_{2,\theta}(T)}$  and the previous arguments, one obtains

$$\begin{aligned} \|M^{-1}u\|_{\mathbb{H}_{2,\theta}^1(T)}^2 &\leq N^2(\varepsilon)\kappa^2 \left( \|Mu_{xx}\|_{\mathbb{L}_{2,\theta}(T)}^2 + \|u_x\|_{\mathbb{L}_{2,\theta}(T)}^2 + \|M^{-1}u\|_{\mathbb{L}_{2,\theta}(T)}^2 \right) \\ &\quad + NC^2 \left( \|u_x\|_{\mathbb{L}_{2,\theta}(T)}^2 + \|M^{-1}u\|_{\mathbb{L}_{2,\theta}(T)}^2 \right) + N \left( \|Mf\|_{\mathbb{L}_{2,\theta}(T)}^2 + \|g\|_{\mathbb{L}_{2,\theta}(T,\ell_2)}^2 \right), \end{aligned}$$

where  $C$  is introduced after (3.39). By using (3.40) we get

$$\|M^{-1}u\|_{\mathbb{H}_{2,\theta}^1(T)}^2 \leq N(N^2(\varepsilon)\kappa^2 + \varepsilon^2)\|M^{-1}u\|_{\mathbb{H}_{2,\theta}^2(T)}^2 + N\left(\|Mf\|_{\mathbb{L}_{2,\theta}(T)}^2 + \|g\|_{\mathbb{L}_{2,\theta}(T,\ell_2)}^2\right).$$

Finally, by substituting this into (3.42) and then choosing  $\varepsilon$  and then  $\kappa_0$  properly, one gets the desired estimate.

**Case 3.**  $\gamma \in \{1, 2, \dots\}$ . Take  $\kappa_0$  from Case 2 and assume  $\kappa \leq \kappa_0$ . Proceed as in Case 2 with  $\varepsilon = 1$ . By (3.34),

$$\|(a_{kr}^{ij} - a_{kr,n}^{ij})\eta_n^2 M u_{x^i x^j}\|_{\mathbb{H}_{2,\theta}^\gamma(T)} \leq N\kappa\|\eta_n M u_{xx}\|_{\mathbb{H}_{2,\theta}^\gamma(T)} + N\|\eta_n M u_{xx}\|_{\mathbb{H}_{2,\theta}^{\gamma-1}(T)},$$

Similarly,

$$\begin{aligned} \|f_n\|_{\mathbb{H}_{2,\theta}^\gamma(T)} + \|g_n\|_{\mathbb{H}_{2,\theta}^{\gamma+1}(T,\ell_2)} &\leq N\kappa\left(\|\eta_n M u_{xx}\|_{\mathbb{H}_{2,\theta}^\gamma(T)} + \|\eta_n u_x\|_{\mathbb{H}_{2,\theta}^{\gamma+1}(T)} + \|\eta_n M^{-1}u\|_{\mathbb{H}_{2,\theta}^{\gamma+1}(T)}\right) \\ &+ N\left(\|\eta_n M u_{xx}\|_{\mathbb{H}_{2,\theta}^{\gamma-1}(T)} + \|\eta_n u_x\|_{\mathbb{H}_{2,\theta}^\gamma(T)} + \|\eta_n M^{-1}u\|_{\mathbb{H}_{2,\theta}^\gamma(T)}\right). \end{aligned}$$

This easily leads to

$$\|M^{-1}u\|_{\mathbb{H}_{2,\theta}^{\gamma+2}(T)} \leq N_2\kappa\|M^{-1}u\|_{\mathbb{H}_{2,\theta}^{\gamma+2}(T)} + N\|M^{-1}u\|_{\mathbb{H}_{2,\theta}^{\gamma+1}(T)} + N\|Mf\|_{\mathbb{H}_{2,\theta}^\gamma(T)} + N\|g\|_{\mathbb{H}_{2,\theta}^{\gamma+1}(T,\ell_2)}.$$

Now additionally assume  $N_2\kappa \leq 1/4$ . Then it is enough to use the interpolation inequality ([11], Theorem 2.6)

$$\|M^{-1}u\|_{H_{2,\theta}^{\gamma+1}} \leq \varepsilon\|M^{-1}u\|_{H_{2,\theta}^{\gamma+2}} + N(\varepsilon, \gamma)\|M^{-1}u\|_{H_{2,\theta}^2},$$

and the results in Case 2.

**Case 4:**  $\gamma = -1$ . We temporarily assume that (3.27) holds with  $\gamma = 1$ . In this case we prove the theorem directly without depending on an a priori estimate. Take  $\kappa_0$  which corresponds to the case  $\gamma = 0$ . Assume  $\kappa \leq \kappa_0$ , then the operator  $\mathcal{R}$  which maps the couples  $(f, g) \in M^{-1}\mathbb{L}_{2,\theta}(T) \times \mathbb{H}_{2,\theta}^1(T, \ell_2)$  into the solutions  $u \in \mathfrak{H}_{2,\theta}^2(T)$  of the problem (1.1) defined on  $\Omega \times [0, T] \times \mathbb{R}_+^d$  with zero initial data is well-defined and bounded.

Now take  $(f, g) \in M^{-1}\mathbb{H}_{2,\theta}^{-1}(T) \times \mathbb{L}_{2,\theta}(T, \ell_2)$ . By Corollary 2.12 in [13] we have the following representations

$$f = MD_\ell f^\ell, \quad g = MD_\ell g^\ell, \quad (3.43)$$

where  $f^\ell = (f^{\ell,1}, \dots, f^{\ell,d_1}) \in M^{-1}\mathbb{L}_{2,\theta}(T)$ ,  $g^\ell = (g^{\ell,1}, \dots, g^{\ell,d_1}) \in \mathbb{H}_{2,\theta}^1(T, \ell_2)$ ,  $\ell = 1, 2, \dots, d$  and

$$\sum_{\ell=1}^d \|Mf^\ell\|_{\mathbb{L}_{2,\theta}(T)} \leq N\|Mf\|_{\mathbb{H}_{2,\theta}^{-1}(T)}, \quad \sum_{\ell=1}^d \|g^\ell\|_{\mathbb{H}_{2,\theta}^1(T,\ell_2)} \leq N\|g\|_{\mathbb{L}_{2,\theta}(T,\ell_2)}. \quad (3.44)$$

Next denote  $v^\ell = (v^{\ell,1}, \dots, v^{\ell,d_1}) = \mathcal{R}(f^\ell, g^\ell)$  and  $\bar{v} = (\bar{v}^1, \dots, \bar{v}^{d_1}) = \sum_{\ell=1}^d MD_\ell v^\ell$ . Then by (3.40),  $\bar{v}$  is in  $M\mathbb{H}_{2,\theta}^1(T)$  and satisfies

$$d\bar{v}^k = (a_{kr}^{ij} \bar{v}_{x^i x^j}^r + b_{kr}^i \bar{v}_{x^i}^r + c_{kr} \bar{v}^r + f^k + \bar{f}^k) dt + (\sigma_{kr,m}^i \bar{v}_{x^i}^r + \nu_{kr,m} \bar{v}^r + g_m^k + \bar{g}_m^k) dw_t^m,$$

where

$$\begin{aligned}\bar{f}^k &= (MD_\ell a_{kr}^{ij})v_{x^i x^j}^{\ell, r} - 2a_{kr}^{i1}v_{x^\ell x^i}^{\ell, r} + (M^2 D_\ell b_{kr}^i)M^{-1}v_{x^i}^{\ell, r} - Mb_{kr}^1 M^{-1}v_{x^\ell}^{\ell, r} + (M^3 D_\ell c_{kr})M^{-2}v^\ell, \\ \bar{g}^k &= (MD_\ell \sigma_{kr}^i)v_{x^i}^{\ell, r} - \sigma_{kr}^1 v_{x^\ell}^{\ell, r} + (M^2 D_\ell \nu_{kr})M^{-1}v_{x^\ell}^{\ell, r}.\end{aligned}$$

By assumptions one can easily check that  $|\cdot|_0^{(0)}$ -norm of  $MD_\ell a_{kr}^{ij}$ ,  $M^2 D_\ell b_{kr}^i$ ,  $M^3 D_\ell c_{kr}$  and  $|\cdot|_1^{(0)}$ -norm of  $MD_\ell \sigma_{kr}^i$ ,  $M^2 D_\ell \nu_{kr}$  are finite. Therefore

$$M\bar{f} \in \mathbb{L}_{2,\theta}(T), \quad \bar{g} \in \mathbb{H}_{2,\theta}^1(T, \ell_2).$$

Finally we define  $\bar{u} = \mathcal{R}(\bar{f}, \bar{g})$  and  $u := \bar{v} - \bar{u}$ . Then  $u \in \mathfrak{H}_{2,\theta}^1(T)$  satisfies (1.1) and the a priori estimate follows from the formulas defining  $\bar{u}$  and  $\bar{v}$ .

Next, we prove the uniqueness of solutions. Let  $\kappa \leq \kappa_0$  with  $\kappa_0$  found above for the case  $\gamma = 0$  and assume  $u \in \mathfrak{H}_{2,\theta}^1(T)$  satisfies (1.1) with  $f = 0$ ,  $g^k = 0$  and  $u_0 = 0$ . Since we already have the uniqueness in the space  $\mathfrak{H}_{2,\theta}^2(T)$ , to show  $u \equiv 0$  we only need to show  $u \in \mathfrak{H}_{2,\theta}^2(T)$ . Take  $\eta_n$  from Lemma 3.16 corresponding to  $\varepsilon = 1$ . From (1.1) one can write the system for  $\eta_n u$  for each  $n$  and get

$$\begin{aligned}d(\eta_n u^k) &= \left( a_{kr}^{ij}(\eta_n u^r)_{x^i x^j} + b_{kr}^i(\eta_n u^r)_{x^i} + c_{kr}(\eta_n u^r) + \tilde{f}_n^k \right) dt \\ &\quad + (\sigma_{kr,m}^i(\eta_n u^r)_{x^i} + \nu_{kr,m}(\eta_n u^r) + \tilde{g}_{n,m}^k) dw_t^m,\end{aligned}$$

where

$$\tilde{f}_n^k = -2a_{kr}^{ij}\eta_{nx^i}u_{x^j}^r - (a_{kr}^{ij}\eta_{nx^i x^j} + b_{kr}^i\eta_{nx^i})u^r, \quad \tilde{g}_{n,m}^k = -\sigma_{kr,m}^i(\eta_n)_{x^i}u^r.$$

Since  $u \in M\mathbb{H}_{2,\theta}^1(T)$  and  $\eta_n$  has compact support, we easily have  $(\tilde{f}, \tilde{g}) \in \mathbb{L}_2(T) \times \mathbb{H}_2^1(T, \ell_2)$ . Also the above system will not change if we arbitrarily change  $a_{kr}^{ij}$ ,  $b_{kr}^i$ ,  $c_{kr}$ ,  $\sigma_{kr}^i$ ,  $\nu_{kr}$  outside of the support of  $\eta_n$ . Therefore using Theorem 2.4, one easily concludes that  $\eta_n u \in \mathbb{H}_2^2(T)$  and hence  $M^{-1}\eta_n u \in \mathbb{H}_{2,\theta}^2(T)$ ,  $\eta_n u \in \mathfrak{H}_{2,\theta}^2(T)$ . Then finally by using (3.29) (which we have for  $\gamma = 0$ ) and Lemma 3.15 one obtains  $\|M^{-1}u\|_{\mathbb{H}_{2,\theta}^2(T)} < \infty$ , that is,  $u \in \mathfrak{H}_{2,\theta}^2(T)$ .

**Case 5:**  $\gamma = -1$  with no additional assumptions. To prove the a priori estimate we use the results of Case 3. Fix a non-negative smooth function  $\phi \in C_0^\infty(B_{1/2}(0))$  with a unit integral. Define

$$\bar{\sigma}(x) = \int \sigma(y)(x^1)^{-d}\phi\left(\frac{x-y}{x^1}\right)dy,$$

and define  $\bar{\nu}$  similarly. Observe that

$$|\bar{\sigma} - \sigma| \leq \kappa, \quad |M\bar{\nu}| \leq 2\kappa.$$

Also using the fact  $x^1 \leq 2(x^1 - x^1 z^1) \leq 4x^1$  for  $|z^1| \leq 1/2$ , one can easily check that there is a constant  $N_0 < \infty$  such that

$$|\bar{\sigma}|_2^{(0)} + |\bar{\nu}|_2^{(1)} < N_0.$$

For instance, let  $i, j \geq 2$ , and  $\delta_{1\ell} = 1$  if  $\ell = 1$  and  $\delta_{1\ell} = 0$  otherwise, then

$$x^1 \bar{\sigma}_{x^1}(x) = \int_{|z| \leq 1/2} \sigma(x - x^1 z)[-d\phi(z) + \phi_{x^\ell}(z) \cdot (\delta_{1\ell} - z^\ell)] dz,$$

$$\begin{aligned}
(x^1)^2 \bar{\nu}_{x^1}(x) &= \int_{|z| \leq 1/2} x^1 \nu(x - x^1 z) [-d\phi(z) + \phi_{x^\ell}(z) \cdot (\delta_{1\ell} - z^\ell)] dz, \\
(x^1)^2 \bar{\sigma}_{x^i x^j}(x) &= \int_{|z| \leq 1/2} \sigma(x - x^1 z) \phi_{x^i x^j}(z) dz, \\
(x^1)^3 \bar{\nu}_{x^i x^j}(x) &= \int_{|z| \leq 1/2} x^1 \nu(x - x^1 z) \phi_{x^i x^j}(z) dz,
\end{aligned}$$

and therefore it is obvious that the functions above are bounded. Also, all other cases can be considered similarly.

Take  $(f, g) \in M^{-1}\mathbb{H}_{2,\theta}^{-1}(T) \times \mathbb{L}_{2,\theta}(T, \ell_2)$  and let  $u \in \mathfrak{H}_{2,\theta}^1(T)$  be a solution of (1.1) with zero initial data. Then

$$du^k = (a_{kr}^{ij} u_{x^i x^j}^r + b_{kr}^i u_{x^i}^r + c_{kr} u^r + f^k) dt + (\bar{\sigma}_{kr,m}^i u_{x^i}^r + \bar{\nu}_{kr,m} u^r + \bar{g}_m^k) dw_t^m,$$

where  $\bar{g}^k = g^k + (\sigma_{kr}^i - \bar{\sigma}_{kr}^i) u_{x^i}^r + (\nu_{kr} - \bar{\nu}_{kr}) u^r$ . Note

$$\|\bar{g}\|_{\mathbb{L}_{2,\theta}(T, \ell_2)} \leq \|g\|_{\mathbb{L}_{2,\theta}(T, \ell_2)} + \kappa \|u_x\|_{\mathbb{L}_{2,\theta}(T)} + 3\kappa \|M^{-1}u\|_{\mathbb{L}_{2,\theta}(T)}. \quad (3.45)$$

Thus, by the results of Case 4, if  $\kappa \leq \kappa_0$ , then

$$\begin{aligned}
\|M^{-1}u\|_{\mathbb{H}_{2,\theta}^1(T)} &\leq N \left( \|Mf\|_{\mathbb{H}_{2,\theta}^{-1}(T)} + \|\bar{g}\|_{\mathbb{L}_{2,\theta}(T, \ell_2)} \right) \\
&\leq N_1 \left( \|Mf\|_{\mathbb{H}_{2,\theta}^{-1}(T)} + \|g\|_{\mathbb{L}_{2,\theta}(T, \ell_2)} + \kappa \|M^{-1}u\|_{\mathbb{H}_{2,\theta}^1(T)} \right), \quad (3.46)
\end{aligned}$$

where the second inequality comes from (3.45) and (3.40). Finally we assume

$$\kappa \leq \kappa_0 \wedge (2N_1)^{-1}.$$

Then (3.46) yields

$$\|M^{-1}u\|_{\mathbb{H}_{2,\theta}^1(T)} \leq 2N_1 \left( \|Mf\|_{\mathbb{H}_{2,\theta}^{-1}(T)} + \|g\|_{\mathbb{L}_{2,\theta}(T, \ell_2)} \right).$$

Thus we get the desired result for  $\gamma = -1$ .

**Case 6:**  $\gamma = -2, -3, -4, \dots$  In this case it is enough to repeat the processes in Case 4, but since  $|\gamma| \geq |\gamma + 2|$ , additional smoothness assumption on the coefficients is unnecessary. The theorem is proved.  $\square$

## 4 The system with bounded $C^1$ -domain $\mathcal{O}$

**Assumption 4.1.** The bounded domain  $\mathcal{O}$  is of class  $C_u^1$ . In other words, for any  $x_0 \in \partial\mathcal{O}$ , there exist constants  $r_0, K_0 \in (0, \infty)$  and a one-to-one continuously differentiable mapping  $\Psi$  of  $B_{r_0}(x_0)$  onto a domain  $J \subset \mathbb{R}^d$  such that

- (i)  $J_+ := \Psi(B_{r_0}(x_0) \cap \mathcal{O}) \subset \mathbb{R}_+^d$  and  $\Psi(x_0) = 0$ ;
- (ii)  $\Psi(B_{r_0}(x_0) \cap \partial\mathcal{O}) = J \cap \{y \in \mathbb{R}^d : y^1 = 0\}$ ;
- (iii)  $\|\Psi\|_{C^1(B_{r_0}(x_0))} \leq K_0$  and  $|\Psi^{-1}(y_1) - \Psi^{-1}(y_2)| \leq K_0 |y_1 - y_2|$  for any  $y_i \in J$ ;
- (iv)  $\Psi_x$  is uniformly continuous in for  $B_{r_0}(x_0)$ .



To proceed further we introduce some well known results from [3] and [8] (also, see [17] for the details).

**Lemma 4.2.** *Let the domain  $\mathcal{O}$  be of class  $C_u^1$ . Then*

(i) *there is a bounded real-valued function  $\psi$  defined in  $\bar{\mathcal{O}}$ , the closure of  $\mathcal{O}$ , such that the functions  $\psi(x)$  and  $\rho(x) := \text{dist}(x, \partial\mathcal{O})$  are comparable. In other words,  $N^{-1}\rho(x) \leq \psi(x) \leq N\rho(x)$  with some constant  $N$  independent of  $x$ ,*

(ii) *for any multi-index  $\alpha$ ,*

$$\sup_{\mathcal{O}} \psi^{|\alpha|}(x) |D^\alpha \psi(x)| < \infty. \quad (4.1)$$

Now, we take the Banach spaces introduced in [8] and [19]. Let  $\zeta \in C_0^\infty(\mathbb{R}_+)$  be a function satisfying (3.1). For  $x \in \mathcal{O}$  and  $n \in \mathbb{Z} = \{0, \pm 1, \dots\}$  we define

$$\zeta_n(x) = \zeta(e^n \psi(x)).$$

Then we have  $\sum_n \zeta_n \geq c$  in  $\mathcal{O}$  and

$$\zeta_n \in C_0^\infty(\mathcal{O}), \quad |D^m \zeta_n(x)| \leq N(m) e^{mn}.$$

For  $\theta, \gamma \in \mathbb{R}$ , let  $H_{p,\theta}^\gamma(\mathcal{O})$  be the set of all distributions  $u = (u^1, u^2, \dots, u^{d_1})$  on  $\mathcal{O}$  such that

$$\|u\|_{H_{p,\theta}^\gamma(\mathcal{O})}^p := \sum_{n \in \mathbb{Z}} e^{n\theta} \|\zeta_{-n}(e^n \cdot) u(e^n \cdot)\|_{H_p^\gamma}^p < \infty. \quad (4.2)$$

If  $g = (g^1, g^2, \dots, g^{d_1})$  and each  $g^k$  is an  $\ell_2$ -valued function, then we define

$$\|g\|_{H_{p,\theta}^\gamma(\mathcal{O}, \ell_2)}^p = \sum_{n \in \mathbb{Z}} e^{n\theta} \|\zeta_{-n}(e^n \cdot) g(e^n \cdot)\|_{H_p^\gamma(\ell_2)}^p.$$

It is known (see, for instance, [19]) that up to equivalent norms the space  $H_{p,\theta}^\gamma(\mathcal{O})$  is independent of the choice of  $\zeta$  and  $\psi$ . Moreover if  $\gamma$  is a non-negative integer, then

$$H_{p,\theta}^\gamma(\mathcal{O}) = \{u : u, \psi Du, \dots, \psi^{|\alpha|} D^\alpha u \in L_p(\mathcal{O}, \psi^{\theta-d} dx), |\alpha| \leq \gamma\},$$

$$\|u\|_{H_{p,\theta}^\gamma(\mathcal{O})}^p \sim \sum_{|\alpha| \leq \gamma} \int_{\mathcal{O}} |\psi^{|\alpha|} D^\alpha u(x)|^p \psi^{\theta-d} dx.$$

Denote  $\rho(x, y) = \rho(x) \wedge \rho(y)$  and  $\psi(x, y) = \psi(x) \wedge \psi(y)$ . For  $n \in \mathbb{Z}$ ,  $\mu \in (0, 1]$  and  $k = 0, 1, 2, \dots$ , we define

$$\begin{aligned} |u|_C &= \sup_{\mathcal{O}} |u(x)|, \quad [u]_{C^\mu} = \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\mu}. \\ [u]_k^{(n)} &= [u]_{k,\mathcal{O}}^{(n)} = \sup_{\substack{x \in \mathcal{O} \\ |\beta|=k}} \psi^{k+n}(x) |D^\beta u(x)|, \end{aligned} \quad (4.3)$$

$$[u]_{k+\mu}^{(n)} = [u]_{k+\mu,\mathcal{O}}^{(n)} = \sup_{\substack{x, y \in \mathcal{O} \\ |\beta|=k}} \psi^{k+\mu+n}(x, y) \frac{|D^\beta u(x) - D^\beta u(y)|}{|x - y|^\mu}, \quad (4.4)$$

$$|u|_k^{(n)} = |u|_{k,\mathcal{O}}^{(n)} = \sum_{j=0}^k [u]_{j,\mathcal{O}}^{(n)}, \quad |u|_{k+\mu}^{(n)} = |u|_{k+\mu,\mathcal{O}}^{(n)} = |u|_{k,\mathcal{O}}^{(n)} + [u]_{k+\mu,\mathcal{O}}^{(n)}.$$

Remember that in case  $\mathcal{O} = \mathbb{R}_+^d$ , to define  $|u|_k^{(n)} = |u|_{k,\mathbb{R}_+^d}^{(n)}$ , we used  $\rho(x)(=x^1)$  and  $\rho(x) \wedge \rho(y)$  in place of  $\psi(x)$  and  $\psi(x,y)$  respectively in (4.3) and (4.4).

Below we collect some other properties of the spaces  $H_{p,\theta}^\gamma(\mathcal{O})$ .

**Lemma 4.3.** ([19]) (i) The assertions (i)-(iii) in Lemma 3.1 hold if one formally replace  $M$  and  $H_{p,\theta}^\gamma$  by  $\psi$  and  $H_{p,\theta}^\gamma(\mathcal{O})$ , respectively.

(ii) There is a constant  $N = N(\gamma, |\gamma|_+, p, \theta) > 0$  so that

$$\|af\|_{H_{p,\theta}^\gamma(\mathcal{O})} \leq N|a|_{|\gamma|_+}^{(0)} \|f\|_{H_{p,\theta}^\gamma(\mathcal{O})}.$$

Denote

$$\mathbb{H}_{p,\theta}^\gamma(\mathcal{O}, T) = L_p(\Omega \times [0, T], \mathcal{P}, H_{p,\theta}^\gamma(\mathcal{O})), \quad \mathbb{H}_{p,\theta}^\gamma(\mathcal{O}, T, \ell_2) = L_p(\Omega \times [0, T], \mathcal{P}, H_{p,\theta}^\gamma(\mathcal{O}, \ell_2)),$$

$$U_{p,\theta}^\gamma(\mathcal{O}) = \psi^{1-2/p} L_p(\Omega, \mathcal{F}_0, H_{p,\theta}^{\gamma-2/p}(\mathcal{O})), \quad \mathbb{L}_{p,\theta}(\mathcal{O}, T) = \mathbb{H}_{p,\theta}^0(\mathcal{O}, T).$$

**Definition 4.4.** We say  $u \in \mathfrak{H}_{p,\theta}^{\gamma+2}(\mathcal{O}, T)$  if  $u = (u^1, \dots, u^{d_1}) \in \psi \mathbb{H}_{p,\theta}^{\gamma+2}(\mathcal{O}, T)$ ,  $u(0, \cdot) \in U_{p,\theta}^{\gamma+2}(\mathcal{O})$  and for some  $f \in \psi^{-1} \mathbb{H}_{p,\theta}^\gamma(\mathcal{O}, T)$ ,  $g \in \mathbb{H}_{p,\theta}^{\gamma+1}(\mathcal{O}, T, \ell_2)$ ,

$$du = f dt + g_m dw_t^m,$$

in the sense of distributions. The norm in  $\mathfrak{H}_{p,\theta}^{\gamma+2}(\mathcal{O}, T)$  is defined by

$$\|u\|_{\mathfrak{H}_{p,\theta}^{\gamma+2}(\mathcal{O}, T)} = \|\psi^{-1}u\|_{\mathbb{H}_{p,\theta}^{\gamma+2}(\mathcal{O}, T)} + \|\psi f\|_{\mathbb{H}_{p,\theta}^\gamma(\mathcal{O}, T)} + \|g\|_{\mathbb{H}_{p,\theta}^{\gamma+1}(\mathcal{O}, T)} + \|u(0, \cdot)\|_{U_{p,\theta}^{\gamma+2}(\mathcal{O})}.$$

The following result is due to N.V. Krylov (see [10] and [5]).

**Lemma 4.5.** Let  $p \geq 2$ . The space  $\mathfrak{H}_{p,\theta}^{\gamma+2}(T)$  is a Banach space and there exists a constant  $c = c(d, p, \theta, \gamma, T)$  such that

$$\mathbb{E} \sup_{t \leq T} \|u(t)\|_{H_{p,\theta}^{\gamma+1}(\mathcal{O})}^p \leq c \|u\|_{\mathfrak{H}_{p,\theta}^{\gamma+2}(\mathcal{O}, T)}^p.$$

In particular, for any  $t \leq T$ ,

$$\|u\|_{\mathbb{H}_{p,\theta}^{\gamma+1}(\mathcal{O}, t)}^p \leq c \int_0^t \|u\|_{\mathfrak{H}_{p,\theta}^{\gamma+2}(\mathcal{O}, s)}^p ds.$$

**Assumption 4.6.** (i) The functions  $a_{kr}^{ij}(t, \cdot), \sigma_{kr}^i(t, \cdot)$  are **point-wise continuous** in  $\mathcal{O}$ . That is, for any  $\varepsilon > 0$  and  $x \in \mathcal{O}$ , there exists  $\delta = \delta(\varepsilon, x)$  such that

$$|a_{kr}^{ij}(t, x) - a_{kr}^{ij}(t, y)| + |\sigma_{kr}^i(t, x) - \sigma_{kr}^i(t, y)|_{\ell_2} < \varepsilon$$

whenever  $x, y \in \mathcal{O}$  and  $|x - y| < \delta$ .

(ii) There is a control on the behavior of  $a_{kr}^{ij}, b_{kr}^i, c_{kr}, \sigma_{kr}^i$  and  $\nu_{kr}$  near  $\partial\mathcal{O}$ , namely,

$$\lim_{\substack{\rho(x) \rightarrow 0 \\ x \in \mathcal{O}}} \sup_{y \in \mathcal{O}} \sup_{\substack{t, \omega \\ |x-y| \leq \rho(x,y)}} [|a_{kr}^{ij}(t, x) - a_{kr}^{ij}(t, y)| + |\sigma_{kr}^i(t, x) - \sigma_{kr}^i(t, y)|_{\ell_2}] = 0. \quad (4.5)$$

$$\lim_{\substack{\rho(x) \rightarrow 0 \\ x \in \mathcal{O}}} \sup_{t, \omega} [\rho(x) |b_{kr}^i(t, x)| + \rho^2(x) |c_{kr}(t, x)| + \rho(x) |\nu_{kr}(t, x)|_{\ell_2}] = 0. \quad (4.6)$$

(iii) For any  $t > 0$  and  $\omega \in \Omega$ ,

$$|a_{kr}^{ij}(t, \cdot)|_{|\gamma|_+}^{(0)} + |b_{kr}^i(t, \cdot)|_{|\gamma|_+}^{(1)} + |c_{kr}(t, \cdot)|_{|\gamma|_+}^{(2)} + |\sigma_{kr}^i(t, \cdot)|_{|\gamma+1|_+}^{(0)} + |\nu_{kr}(t, \cdot)|_{|\gamma+1|_+}^{(1)} \leq L.$$

*Remark 4.7.* (i). The condition (4.5) is equivalent to

$$\lim_{\rho(x) \rightarrow 0} \sup_{\omega, t} \left( \text{osc}(a_{kr}^{ij})_{B_{\frac{\rho(x)}{2}}(x)} + \text{osc}(\sigma_{kr}^i)_{B_{\frac{\rho(x)}{2}}(x)} \right) = 0.$$

(ii). It is easy to see that (4.5) is much weaker than uniform continuity condition. For instance, if  $\delta \in (0, 1)$ ,  $d = d_1 = 1$ , and  $\mathcal{O} = \mathbb{R}_+$ , then the function  $a(x)$  equal to  $2 + \sin(|\ln x|^\delta)$  for  $0 < x \leq 1/2$  satisfies (4.5). Indeed, if  $x, y > 0$  and  $|x - y| \leq x \wedge y$ , then

$$|a(x) - a(y)| = |x - y| |a'(\xi)|,$$

where  $\xi$  lies between  $x$  and  $y$ . In addition,  $|x - y| \leq x \wedge y \leq \xi \leq 2(x \wedge y)$ , and  $\xi |a'(\xi)| \leq |\ln[2(x \wedge y)]|^{\delta-1} \rightarrow 0$  as  $x \wedge y \rightarrow 0$ .

(iii). We observe that (4.6) allows the coefficients  $b_{kr}^i, c_{kr}$  and  $\nu_{kr}$  to blow up near the boundary at a certain rate. For instance, it holds if

$$|b_{kr}^i| \leq N\rho^{-1+\varepsilon}, \quad |c_{kr}| \leq N\rho^{-2+\varepsilon}, \quad |\nu_{kr}|_{\ell_2} \leq N\rho^{-1+\varepsilon}$$

for some constants  $N, \varepsilon > 0$ .

Here is the main result of this article.

**Theorem 4.8.** *Let Assumptions 2.2, 3.10 and 4.6 hold. Then for any  $f \in \psi^{-1}\mathbb{H}_{2,\theta}^\gamma(\mathcal{O}, T)$ ,  $g \in \mathbb{H}_{2,\theta}^{\gamma+1}(\mathcal{O}, T, \ell_2)$ ,  $u_0 \in U_{2,\theta}^{\gamma+2}(\mathcal{O})$ , the problem (1.1) on  $\Omega \times [0, T] \times \mathcal{O}$  admits a unique solution  $u = (u^1, \dots, u^{d_1}) \in \mathfrak{H}_{2,\theta}^{\gamma+2}(\mathcal{O}, T)$ , and for this solution*

$$\|\psi^{-1}u\|_{\mathbb{H}_{2,\theta}^{\gamma+2}(\mathcal{O}, T)} \leq c \left( \|\psi f\|_{\mathbb{H}_{2,\theta}^\gamma(\mathcal{O}, T)} + \|g\|_{\mathbb{H}_{2,\theta}^{\gamma+1}(\mathcal{O}, T, \ell_2)} + \|u_0\|_{U_{2,\theta}^{\gamma+2}(\mathcal{O})} \right), \quad (4.7)$$

where  $c = c(d, d_1, \delta, \theta, K, L, T)$ .

*Remark 4.9.* By inspecting the proofs carefully, one can check that the above theorem hold true even if  $\mathcal{O}$  is not bounded.

*Proof.* Since the theorem was already proved for single equations ([6]), as in the proof of Theorem 2.3, we only need to establish the a priori estimate (4.7) assuming that a solution  $u \in \mathfrak{H}_{2,\theta}^{\gamma+2}(\mathcal{O}, T)$  already exists. As usual, we assume  $u_0 = 0$ .

Let  $x_0 \in \partial\mathcal{O}$  and  $\Psi$  be a function from Assumption 4.1. In [8] it is shown that  $\Psi$  can be chosen in such a way that for any non-negative integer  $n$

$$|\Psi_x|_{n, B_{r_0}(x_0) \cap \mathcal{O}}^{(0)} + |\Psi_x^{-1}|_{n, J_+}^{(0)} < N(n) < \infty \quad (4.8)$$

and

$$\rho(x)\Psi_{xx}(x) \rightarrow 0 \quad \text{as } x \in B_{r_0}(x_0) \cap \mathcal{O}, \text{ and } \rho(x) \rightarrow 0, \quad (4.9)$$

where the constants  $N(n)$  and the convergence in (4.9) are independent of  $x_0$ .

Define  $r = r_0/K_0$  and fix smooth functions  $\eta \in C_0^\infty(B_1(0))$ ,  $\varphi \in C^\infty(\mathbb{R})$  such that  $0 \leq \eta, \varphi \leq 1$ , and  $\eta = 1$  in  $B_{r/2}(0)$ ,  $\varphi(t) = 1$  for  $t \leq -3$ , and  $\varphi(t) = 0$  for  $t \geq -1$  and  $0 \geq \varphi' \geq -1$ . Observe that  $\Psi(B_{r_0}(x_0))$  contains  $B_r(0)$ . For  $n = 1, 2, \dots$ ,  $t > 0$  and  $x \in \mathbb{R}_+^d$  we introduce  $\varphi_n(x) = \varphi(n^{-1} \ln x^1)$ ,

$$a^{ij,n} = (a_{kr}^{ij,n}) := \tilde{a}^{ij}\eta(x)\varphi_n + \delta^{ij}(1 - \eta\varphi_n)I, \quad b^{i,n} = (b_{kr}^{i,n}) := \tilde{b}^i\eta\varphi_n, \quad c^n = (c_{kr}^n) := \tilde{c}\eta\varphi_n,$$

$$\sigma^{i,n} = (\sigma_{kr}^{i,n}) := \tilde{\sigma}\eta\varphi_n, \quad \nu^n = (\nu_{kr}^n) := \tilde{\nu}\eta\varphi_n,$$

where

$$\tilde{a}^{ij}(t, x) = \bar{a}^{ij}(t, \Psi^{-1}(x)), \quad \tilde{b}^i(t, x) = \bar{b}^i(t, \Psi^{-1}(x)),$$

$$\tilde{\sigma}^i(t, x) = \bar{\sigma}^i(t, \Psi^{-1}(x)), \quad \bar{a}^{ij} = \sum_{s,t=1}^d a^{st}\Psi_{x^s}^i\Psi_{x^t}^j,$$

$$\bar{b}^i = \sum_{s,t} a^{st}\Psi_{x^s x^t}^i + \sum_{\ell} b^\ell \Psi_{x^\ell}^i, \quad \bar{\sigma}^i = \sum_s \sigma^s \Psi_{x^s}^i,$$

$$\tilde{c}(t, x) = c(t, \Psi^{-1}(x)), \quad \tilde{\nu}(t, x) = \nu(t, \Psi^{-1}(x)).$$

Using Lemma 3.4 of [8], one can easily check that there is a constant  $L'$  independent of  $n$  and  $x_0$  so that

$$|a^{ij,n}(t, \cdot)|_{|\gamma|_+}^{(0)} + |b^{i,n}(t, \cdot)|_{|\gamma|_+}^{(1)} + |c^n(t, \cdot)|_{|\gamma|_+}^{(2)} + |\sigma_{kr}^{i,n}(t, \cdot)|_{|\gamma+1|_+}^{(0)} + |\nu_{kr}^n(t, \cdot)|_{|\gamma+1|_+}^{(1)} \leq L'. \quad (4.10)$$

Take  $\kappa_0$  from Theorem 3.12 corresponding to  $d, \theta, \delta, K, \gamma$  and  $L'$ . Observe that  $\varphi(m^{-1} \ln x^1) = 0$  for  $x^1 \geq e^{-m}$  and  $|\varphi(m^{-1} \ln x^1) - \varphi(m^{-1} \ln y^1)| \leq m^{-1}$  if  $|x^1 - y^1| \leq x^1 \wedge y^1$ . Also we easily see that (4.9) implies  $x^1 \Psi_{xx}(\Psi^{-1}(x)) \rightarrow 0$  as  $x^1 \rightarrow 0$ . Using these facts, (4.5) and (4.6), one can find and fix  $n > 0$  independent of  $x_0$  such that

$$\begin{aligned} & |a_{kr}^{ij,n}(t, x) - a_{kr}^{ij,n}(t, y)| + |\sigma_{kr}^{i,n}(t, x) - \sigma_{kr}^{i,n}(t, y)|_{\ell_2} + x^1 |b_{kr}^{i,n}(t, x)| \\ & + (x^1)^2 |c_{kr}^n(t, x)| + x^1 |\nu_{kr}^n(t, x)|_{\ell_2} \leq \kappa_0, \end{aligned} \quad (4.11)$$

whenever  $t > 0, x, y \in \mathbb{R}_+^d$  and  $|x - y| \leq x^1 \wedge y^1$ . Now we fix a  $\rho_0 < r_0$  such that

$$\Psi(B_{\rho_0}(x_0)) \subset B_{r/2}(0) \cap \{x : x^1 \leq e^{-3n}\}. \quad (4.12)$$

Let  $\xi$  be a smooth function with support in  $B_{\rho_0}(x_0)$  and denote  $v := (\xi u)(\Psi^{-1})$  and continue  $v$  as zero in  $\mathbb{R}_+^d \setminus \Psi(B_{\rho_0}(x_0))$ . Since  $\eta\varphi_n = 1$  on  $\Psi(B_{\rho_0}(x_0))$ , the function  $v$  satisfies

$$dv^k = (a_{kr}^{ij,n} v_{x^i x^j}^r + b_{kr}^{i,n} v_{x^i}^r + c_{kr}^n v + \hat{f}^k) dt + (\sigma_{kr,m}^{i,n} v_{x^i}^r + \nu_{kr,m}^n v + \hat{g}_m^k) dw_t^m,$$

where

$$\hat{f}^k = \tilde{f}(\Psi^{-1}), \quad \tilde{f} = -2a_{kr}^{ij} u_{x^i}^r \xi_{x^j} - a_{kr}^{ij} u^r \xi_{x^i x^j} - b_{kr}^i u^r \xi_{x^i} + \xi f^k,$$

$$\hat{g} = \tilde{g}(\Psi^{-1}), \quad \tilde{g}^k = -\sigma_{kr}^i u^r \xi_{x^i} + \xi g^k.$$

Next, we observe that by Lemma 4.2 and Theorem 3.2 in [19] (or see [8]) for any  $\nu, \alpha \in \mathbb{R}$  and  $h \in \psi^{-\alpha} H_{p,\theta}^\nu(\mathcal{O})$  with support in  $B_{\rho_0}(x_0)$  we have

$$\|\psi^\alpha h\|_{H_{p,\theta}^\nu(\mathcal{O})} \sim \|M^\alpha h(\Psi^{-1})\|_{H_{p,\theta}^\nu}. \quad (4.13)$$

Therefore, we conclude that  $v \in \mathfrak{H}_{2,\theta}^{\gamma+2}(T)$ . Hence, by Theorem 3.12 we get for any  $t \leq T$

$$\|M^{-1}v\|_{\mathbb{H}_{2,\theta}^{\gamma+2}(t)} \leq N \left( \|M\hat{f}\|_{\mathbb{H}_{2,\theta}^\gamma(t)} + \|\hat{g}\|_{\mathbb{H}_{2,\theta}^{\gamma+1}(t,\ell_2)} + \|u_0(\Psi^{-1})\zeta(\Psi^{-1})\|_{U_{2,\theta}^{\gamma+2}} \right).$$

By using (4.13) again, we obtain

$$\begin{aligned} & \|\psi^{-1}u\zeta\|_{\mathbb{H}_{2,\theta}^{\gamma+2}(\mathcal{O},t)} \\ & \leq N \|a\xi_x \psi u_x\|_{\mathbb{H}_{2,\theta}^\gamma(\mathcal{O},t)} + N \|a\xi_{xx} \psi u\|_{\mathbb{H}_{2,\theta}^\gamma(\mathcal{O},t)} + N \|\xi_x \psi b u\|_{\mathbb{H}_{2,\theta}^\gamma(\mathcal{O},t)} \\ & \quad + N \|\sigma \xi_x u\|_{\mathbb{H}_{2,\theta}^{\gamma+1}(\mathcal{O},t)} + N \|\xi \psi f\|_{\mathbb{H}_{2,\theta}^\gamma(\mathcal{O},t)} + \|\xi g\|_{\mathbb{H}_{2,\theta}^{\gamma+1}(\mathcal{O},t,\ell_2)} + \|\xi u_0\|_{U_{2,\theta}^{\gamma+2}(\mathcal{O})}. \end{aligned}$$

Remembering that  $\rho$  and  $\psi$  are comparable in  $\mathcal{O}$ , one can easily check that the functions

$$|\xi_x a(t, \cdot)|_{|\gamma|_+}^{(0)}, \quad |\xi_{xx} \psi a(t, \cdot)|_{|\gamma|_+}^{(0)}, \quad |\xi_x \psi b(t, \cdot)|_{|\gamma|_+}^{(0)}, \quad |\xi_x \sigma(t, \cdot)|_{|\gamma+1|_+}^{(0)}$$

are bounded on  $\Omega \times [0, T]$ . Then one concludes

$$\begin{aligned} & \|\psi^{-1}u\xi\|_{\mathbb{H}_{2,\theta}^{\gamma+2}(\mathcal{O},t)} \\ & \leq N \|\psi u_x\|_{\mathbb{H}_{2,\theta}^\gamma(\mathcal{O},t)} + N \|u\|_{\mathbb{H}_{2,\theta}^\gamma(\mathcal{O},t)} + N \|\psi f\|_{\mathbb{H}_{2,\theta}^\gamma(\mathcal{O},t)} + \|g\|_{\mathbb{H}_{2,\theta}^{\gamma+1}(\mathcal{O},t,\ell_2)} + N \|u_0\|_{U_{2,\theta}^{\gamma+2}(\mathcal{O})}. \end{aligned}$$

Note that the above constants  $\rho_0, m, L', N$  are independent of  $x_0$ . Therefore, to estimate the norm  $\|\psi^{-1}u\|_{\mathbb{H}_{2,\theta}^{\gamma+2}(\mathcal{O},t)}$ , one introduces a partition of unity  $\xi_{(i)}, i = 0, 1, 2, \dots, N$  such that  $\xi_{(0)} \in C_0^\infty(\mathcal{O})$  and  $\xi_{(i)} \in C_0^\infty(B_{\rho_0}(x_i))$ ,  $x_i \in \partial\mathcal{O}$  for  $i \geq 1$ . Then one estimates  $\|\psi^{-1}u\xi_{(0)}\|_{\mathbb{H}_{2,\theta}^{\gamma+2}(\mathcal{O},t)}$  using Theorem 2.4 and the other norms as above. We only mention that since  $\psi^{-1}u\xi_{(0)}$  has compact support in  $\mathcal{O}$ ,

$$\|\psi^{-1}u\xi_{(0)}\|_{\mathbb{H}_{2,\theta}^{\gamma+2}(\mathcal{O},t)} \sim \|u\xi_{(0)}\|_{\mathbb{H}_{2,\theta}^{\gamma+2}(\mathcal{O},t)} \sim \|u\xi_{(0)}\|_{\mathbb{H}_2^{\gamma+2}(\mathbb{R}^d,t)}.$$

By summing up those estimates one gets

$$\begin{aligned} & \|\psi^{-1}u\|_{\mathbb{H}_{2,\theta}^{\gamma+2}(\mathcal{O},t)} \\ & \leq N \|\psi u_x\|_{\mathbb{H}_{2,\theta}^\gamma(\mathcal{O},t)} + N \|u\|_{\mathbb{H}_{2,\theta}^\gamma(\mathcal{O},t)} + N \|\psi f\|_{\mathbb{H}_{2,\theta}^\gamma(\mathcal{O},t)} + N \|g\|_{\mathbb{H}_{p,\theta}^{\gamma+1}(\mathcal{O},t,\ell_2)} + N \|u_0\|_{U_{2,\theta}^{\gamma+2}(\mathcal{O})}. \end{aligned}$$

By this and the inequality

$$\|\psi u_x\|_{H_{2,\theta}^\gamma(\mathcal{O})} \leq N \|u\|_{H_{2,\theta}^{\gamma+1}(\mathcal{O})},$$

we get for each  $t \leq T$ ,

$$\|u\|_{\mathfrak{H}_{2,\theta}^{\gamma+2}(\mathcal{O},t)}^2 \leq N \|u\|_{\mathbb{H}_{2,\theta}^{\gamma+1}(\mathcal{O},t)}^2 + N \left( \|\psi f\|_{\mathbb{H}_{2,\theta}^\gamma(\mathcal{O},T)}^2 + \|g\|_{\mathbb{H}_{2,\theta}^{\gamma+1}(\mathcal{O},T,\ell_2)}^2 + \|u_0\|_{U_{2,\theta}^{\gamma+2}(\mathcal{O})}^2 \right). \quad (4.14)$$

Now the a priori estimate follows from Lemma 4.5 and Gronwall's inequality. The theorem is proved.  $\square$

## References

- [1] Flandoli, F. (1990). *Dirichlet boundary value problem for stochastic parabolic equations: compatibility relation and regularity of solutions*, Stochastics Stochastics Rep. **29**, no. 3, 331-357.
- [2] Funaki, T. (1983). *Random motion of strings and related stochastic evolution equations*, Nagoya Math. J. **89**, 129-193.
- [3] Gilbarg, D., Hörmander, L. (1980). *Intermediate Schauder estimates*, Archive Rational Mech. Anal., **74**, no. 4, 297-318.
- [4] Gilbarg, D., Trudinger, N.S. (1983). *Elliptic partial differential equations of second order*, 2d ed., Springer Verlag, Berlin.
- [5] Kim, K. (2004).  *$L_q(L_p)$  theory and Hölder estimates for parabolic SPDE*, Stochastic processes and their applications, **114**, no. 2, 313-330.
- [6] Kim, K. (2004). *On stochastic partial differential equations with variable coefficients in  $C^1$  domains*, Stochastic processes and their applications, **112**, no. 2, 261-283.
- [7] Kim, K., Krylov, N.V. (2004). *On SPDEs with variable coefficients in one space dimension*, Potential Anal, **21**, no. 3, 203-239.
- [8] Kim, K., Krylov, N.V. (2004). *On the Sobolev space theory of parabolic and elliptic equations in  $C^1$  domains*, SIAM J. Math. Anal. **36**, 618-642.
- [9] Krylov, N.V. (2008). *Lectures on Elliptic and Parabolic Equations in Sobolev Spaces*, American Mathematical Society, Providence, RI.
- [10] Krylov, N.V. (2001). *Some properties of traces for stochastic and deterministic parabolic weighted Sobolev spaces*, Journal of Functional Analysis **183**, 1-41.
- [11] Krylov, N.V. (1999). *Some properties of weighted Sobolev space in  $\mathbb{R}_+^d$* , Ann. Scuola Norm. Sup. Pisa Cl. Sci. **4**, no.28, 675-693.
- [12] Krylov, N.V. (1999). *An analytic approach to SPDEs*, pp. 185-242 in Stochastic Partial Differential Equations: Six Perspectives, Mathematical Surveys and Monographs, **64**, AMS, Providence, RI.
- [13] Krylov, N.V. (1999). *Weighted Sobolev spaces and Laplace equations and the heat equations in a half space*, Comm. in PDEs, **23**, no. 9-10, 1611-1653.
- [14] Krylov, N.V. (1994). *A  $W_2^n$ -theory of the Dirichlet problem for SPDEs in general smooth domains*, Probab.Theory Relat.Fields **98**, 389-421.
- [15] Krylov, N.V., Lototsky, S.V. (1999). *A Sobolev space theory of SPDEs with constant coefficients on a half line*, SIAM J. Math. Anal., **30**, no. 2, 298-325.
- [16] Krylov, N.V., Lototsky, S.V. (1999). *A Sobolev space theory of SPDEs with constant coefficients in a half space*, SIAM J. on Math. Anal., **31**, no. 1, 19-33.
- [17] Lapić, S.K. (1994). *On the first-initial boundary value problem for stochastic partial differential equations*, Ph.D. thesis, University of Minnesota, Minneapolis, MN.
- [18] Lototsky, S.V. (1999). *Dirichlet problem for stochastic parabolic equations in smooth domains*, Stochastics and Stochastics Reports, **68**, no. 1-2, 145-175.
- [19] Lototsky, S.V. (2000). *Sobolev spaces with weights in domains and boundary value problems for degenerate elliptic equations*, Methods and Applications of Analysis, **1**, no. 1, 195-204.
- [20] Mikulevicius, R., Rozovskii, B. (2004). *Stochastic Navier-Stokes equations for turbulent flows*, SIAM J. Math. Anal., **35**, no. 5, 1250-1310.

- [21] Mikulevicius, R., Rozovskii, B. (2001). *A note on Krylov's  $L_p$ -theory for systems of SPDEs*, Electron. J. Probab., **6**, no. 12, 1-35.
- [22] Mueller, C., Tribe, R. (2002). *Hitting properties of a random string*, Electron. J. Probab. **7**, no. 10, 1-29.
- [23] Rozovskii, B. (1990). *Stochastic evolution systems*, Kluwer, Dordrecht.
- [24] Triebel, H. (1983). *Theory of function spaces*, Birkhäuser Verlag, Basel-Boston-Stuttgart.